Let $R$ be an associative ring with identity and $M$ be unital non zero right $R$-module. $M$ is called $H$-supplemented module if given any submodule $A$ of $M$ there exist a direct summand submodule $D$ of $M$ such that $M = A + X$ iff $M = D + X$ where $X$ is a submodule of $M$. In this paper we will give a generalization for $H$-supplemented which is called pure-supplemented module. An $R$-module $M$ is called pure-supplemented module if given any submodule $A$ of $M$ there exists a pure submodules $P$ of $M$ such that $M = A + X$ iff $M = P + X$ .

Equivalently, for every submodule $A$ of $M$ there exists a pure submodule $P$ of $M$ such that $A + P \ll P$ and $A + P \ll M$.

**Key words:** Small submodule, Supplemented module, Pure module, lifting module.

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**Abstract.**

Let $R$ be an associative ring with identity and $M$ be unital non zero right $R$-module. $M$ is called $H$-supplemented module if given any submodule $A$ of $M$ there exist a direct summand submodule $D$ of $M$ such that $M = A + X$ iff $M = D + X$ where $X$ is a submodule of $M$. In this paper we will give a generalization for $H$-supplemented which is called pure-supplemented module. An $R$-module $M$ is called pure-supplemented module if given any submodule $A$ of $M$ there exists a pure submodules $P$ of $M$ such that $M = A + X$ iff $M = P + X$ .

Equivalently, for every submodule $A$ of $M$ there exist a pure submodule $P$ of $M$ such that $A + P \ll P$ and $A + P \ll M$.

**Key words:** Small submodule, Supplemented module, Pure module, lifting module.

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**Pure–Supplemented Modules**

**Sahira Mahmood Yasen, Wasan Khalid Hasan**

Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.

sahira.mahmood@gmail.com.
Introduction:

Let $R$ be an associative ring with identity and $M$ be a non zero unital right $R$-module. A submodule $N$ of $M$ is called a small submodule of $M$, denoted by $N \ll M$, if $N + L \neq M$ for any proper submodule $L$ of $M$ [1]. Let $U$ be a submodule of $M$, a submodule $V$ of $M$ is called supplement of $U$ if $V$ is minimal element in the set of submodules $L \leq M$ with $U + L = M$ equivalently $U + V = M$ and $U \cap V \ll V$. An $R$-module $M$ is called supplemented if every submodule of $M$ has supplement in $M$ [2].

$M$ is called $H$–supplemented module if given any submodule $A$ of $M$ there exist a direct summand $D$ of $M$ such that $M = A + X$ iff $M = D + X$.

**Definition 1.2:** [3] Let $M$ be an $R$-module. $P$ is called pure submodule of $M$ if $KM \cap P = KP$ for every ideal $K$ in $R$.

**Remarks 1.3:** [3]

1) Any direct summand submodule is pure submodule in $M$.
2) If $H \leq M$ and $K \leq H$ such that $H$ is pure in $M$ and $K$ is pure in $H$ then $K$ is pure in $M$.
3) If $A$ is pure submodule of $M$ and $K$ is pure submodule of $N$ then $A \oplus K$ is pure of $M \oplus N$.

As a generalization of $H$-supplemented module, we introduce the pure-supplemented module.

**Definition 1.4:** Let $M$ be a module. $M$ is called pure–supplemented module if given any submodule $A$ of $M$ there exists a pure submodule $P$ of $M$ such that $M = A + X$ iff $M = P + X$.

Since every direct summand submodule is pure then it is clear that each $H$–supplemented module is pure–supplemented.

**Remark 1.5:**

1) Every hollow module is pure-supplemented module.
2) Every lifting module is pure-supplemented module.
3) Every $P$-supplemented is weakly supplemented.

**Proof:**

1) Since every hollow module is $H$-supplemented module then is pure-supplemented module.
2) Let $A$ be submodule of $M$ there exists $K \leq A$, $M = N \oplus K$ where $N \leq M$ and $N \cap A \ll M$ then $M = A + N$ iff $M = K + N$ where $K$ is pure since $K$ is direct summand submodule.
3) Let $N$ be submodule of $M$, such that $N + X = M$, to show that $N \cap X \ll M$. Let $(N \cap X) + L = M$, since $M$ pure–supplemented there exists a pure submodule $P$ of $M$ such that $N \cap X + L = M$, iff $M = P + L$ iff...
N(1) X+P ≤ L then N(1) X ≤ L hence L=M and

In this section we introduce the pure – lifting module as a generalization of lifting module.

Definition 1.6: Let M be a module . M is called pure– lifting module if for every submodule A of M there exists a pure submodule P of M , P≤ A such that M = P+X with A∩X<<X.

It is clear that every lifting and simesimple module are pure– lifting module .

Theorem 1.7: The following are equivalent for an R- module M .

1) M is pure– lifting module
2)Every submodule N of M can be written as N=A+S where A is pure in M and S << M.
3)For every submodule N of M there exists a pure submodule A of N such that M= A +K and \( \frac{N}{A} << \frac{M}{A} \).


2→3) Let N be a submodule of M by (2) N=A+S where A is pure in M and S << M suppose \( \frac{M}{A} = \frac{N}{A} + \frac{L}{A} \).Then \( \frac{M}{A} = \frac{A+S}{A} + \frac{L}{A} \), Thus A+S+L=M, by(2) since S << M then A+L=M.

3→1) Let N be submodule of M ,there exists a pure submodule A of Nsuch that M= A +K and \( \frac{N}{A} << \frac{M}{A} \) to prove that N∩K<<K. Suppose that N∩K+B=K where B≤Kthen M=A+K=N∩K+B thus \( \frac{M}{A} = \frac{A+(N \cap K) + B}{A} = \frac{N \cap K + A + A + B}{A} = \frac{N + A + B}{A} \),since \( \frac{N}{A} << \frac{M}{A} \) then \( \frac{A + B}{A} = \frac{M}{A} \) thus A+B=M then and hence B=K thus N∩K<<K.

Proposition 1.8: Every pure – lifting is pure- supplemented module .

Proof :Let M be pure – lifting and A be a submodule of M suppose that M=A+Y then M=K+L where K≤ A and K pure in M and A∩L<<M, now A= A∩M = A∩(K+L)=K+ A∩L then

Let M= A+X =K+ A∩L +X since A∩L<<M then M=K+X thus M=A+X since K≤L then M is pure- supplemented module .

Proposition 1.9: Let M be an R- module M is pure– supplemented module iff for every submodule A of M there exist a pure submodule P of M such that \( \frac{A+P}{P} << \frac{M}{P} \) and \( \frac{A+P}{A} << \frac{M}{A} \).

Proof : (⇒) Let M be a pure- supplemented module .and A≤ M then there exist a pure submodule P such that M = A+X iff M= P+X . suppose that \( \frac{A+P}{P} + \frac{L}{P} = \frac{M}{P} \) then \( \frac{A+L}{P} = \frac{M}{P} \) thus \( \frac{A+L}{P} = \frac{M}{P} \) then

L=M .M is pure - supplemented ,A+L=M=P+X, P≤L ,then A+X≤L, then M≤L thus \( \frac{L}{P} = \frac{M}{P} \) therefore \( \frac{A+P}{P} << \frac{M}{P} \)

similarly \( \frac{A+P}{A} << \frac{M}{A} \).

(⇐) Let A be submodule of M ,then there exists a pure submodule P of M suchthat \( \frac{A+P}{P} << \frac{M}{P} \) and \( \frac{A+P}{A} << \frac{M}{A} \).If M=A+X then \( \frac{A+X}{P} = \frac{M}{P} \) then \( \frac{A+P}{P} + \frac{X+P}{P} \)

but \( \frac{A+P}{P} << \frac{M}{P} \).Then \( \frac{M}{P} - \frac{X+P}{P} \) thus M=X+P .In the same way one can show that if M= X+P then M=A+X.

Proposition 1.10 Let M be pure– supplemented module and A be a submodule of M .If for every pure submodule P of M ,
\[ \frac{A + P}{A} \] is pure in \[ \frac{M}{A} \] then \[ \frac{M}{A} \] is pure–supplemented module.

**Proof:** Let \( \frac{N}{A} \leq \frac{M}{A} \), and let \[ \frac{M}{A} = \frac{N}{A} + \frac{X}{A} \]
where \( A \leq X \) then \( M = N + X \) iff \( M = P + X \) where \( P \) is pure in \( M \), \((M\) pure–supplemented). Then \[ \frac{M}{A} = \frac{P + X}{A} = \frac{P + A}{A} + \frac{X}{A} \]
by assumption \( \frac{A + P}{A} \) is pure in \( \frac{M}{A} \) hence \( \frac{M}{A} \) is pure – supplemented.

Recall that a submodule \( A \) of \( R \)-module \( M \) is called fully invariant if for every \( f \in \text{End}_R(M) \), \( f(X) \subseteq X \). A module \( M \) is called distributive iff for every submodules \( K, L, N \), of \( M \) we have \( N + (K \cap L) = (N + K) \cap (N + L) \) or \( N \cap (K + L) = (N \cap K) + (N \cap L) \).

**Corollary 1.11:** Let \( M \) be a distributive pure–supplemented module then \( \frac{M}{M} \) is pure–supplemented module for every submodule \( A \) of \( M \).

**Proof:** Let \( D \) be direct summand of \( M \), then \( M = D \oplus K \) for some \( K \) submodule of \( M \).

\[ \frac{M}{A} = \frac{D + A}{A} + \frac{K + A}{A} \]
and \( A = A + (D \cap K) = (A + D) \cap (A + K) \) (\( M \) is distributive) then \( \frac{M}{A} = \frac{D + A}{A} \oplus \frac{K + A}{A} \) hence \( \frac{D + A}{A} \) is direct summand of \( \frac{M}{A} \),
then is pure in \( \frac{M}{A} \) thus by proposition (1.10) we get \( \frac{M}{A} \) is pure–supplemented.

**Corollary 1.12:** Let \( A \) be a submodule of \( M \) and \( eA \subseteq A \) for all \( e^2 = e \in \text{End}_R(M) \) then \( \frac{M}{A} \) is pure–supplemented. In particular for every fully invariant submodule \( Y \) of \( M \), \( \frac{M}{Y} \) is pure – supplemented.

**Proof:** Let \( D \) is a direct summand of \( M \) consider the projection map \( e : M \rightarrow D \) then \( e^2 = e \in \text{End}_R(M) \), \( eA \subseteq A \) and hence \( eA = A \cap D \). Since \( D \) is a direct summand of \( M \) then \( M = D \oplus K \), \( K \leq M \) hence \( A = (A \cap D) \oplus (A \cap K) \) now \( \frac{D + A}{A} = \frac{D \oplus (A \cap K)}{A} \) and \( K + A = \frac{K \oplus (A \cap D)}{A} \) hence \( \frac{K + A}{A} \) is direct summand of \( \frac{M}{A} \) then is pure in \( \frac{M}{A} \) and by (prop. 1.10) \( \frac{M}{A} \) is pure–supplemented.

**2–Completely pure–supplemented Modules**

We call a module \( M \) is completely pure–supplemented module if every direct summand of \( M \) is pure–supplemented.

**Proposition 2.1:** Every lifting is completely pure–supplemented module.

**Proof:** Let \( M = D \oplus K \) for some \( K \) submodule of \( M \).

\[ \frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A} \]
Then \( \frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A} \) and \( A = A \cap (D \cap K) = (A \cap D) \cap (A \cap K) \) then \( \frac{M}{A} = \frac{D}{A} \oplus \frac{K}{A} \).

Hence \( \frac{K + A}{A} \) is direct summand of \( \frac{M}{A} \) then is pure in \( \frac{M}{A} \) and by (prop. 1.10) \( \frac{M}{A} \) is pure–supplemented.

**Proposition 2.2:**

Let \( M \) be pure–supplemented module and \( M \) has \( \text{PSP} \) then \( M \) is completely pure–supplemented module.

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**Proof:** Let N be a direct summand of M. We show that N is pure-supplemented. \( M = N \oplus K \) and \( K \leq M \) assume P is pure in M. Then by assumption \( N + P \) pure in M, \( M = N + P \) is pure-supplemented (prop. 1.10), but \( M = N + P \). Therefore, N is pure-supplemented module.

**Proposition 2.3:** If an R-module M has PSP and \( M = M_1 \oplus M_2 \) is duo module, then M is pure-supplemented iff \( M_1 \) and \( M_2 \) are pure-supplemented modules.

**Proof:** \( \Rightarrow \) Since \( M_1 \) and \( M_2 \) are fully invariant submodules, hence \( M_1 \) and \( M_2 \) are pure-supplemented modules (coro. 1.12).

\( \Leftarrow \) Assume \( M_1 \) and \( M_2 \) are pure-supplemented modules and let \( L \leq M \) then there exist a pure submodule \( P_1 \) of M such that \( M_1 = (L \cap M_1) + X \) iff \( M_1 = P_1 + X \) for any submodule \( X \) of \( M_1 \). And there exist a pure submodule \( P_2 \) of M such that \( M_2 = (L \cap M_2) + Y \) iff \( M_2 = P_2 + Y \) for any submodule \( Y \) of \( M_2 \).

Claim \( M = (P_1 \oplus P_2) + Z \) iff \( M = L + Z \) for any submodule Z of M. Assume \( M = (P_1 + P_2) + Z \) then \( M_1 = P_1 + (M_1 \cap (P_2 + Z)) = (L \cap M_1) + (M_1 \cap (P_2 + Z)) = M_1 \cap [(L \cap M_1) + (P_2 + Z)] \) then \( M_1 \leq (L \cap M_1) + Z \), since \( m_1 \in M \).

\( m_1 = x + a_2 + z \) where \( a_2 \in P_2, x \in L \cap M_1, z \in Z \) since \( Z = (Z \cap M_1) \oplus (Z \cap M_2) \) then \( z = z_1 + z_2 \) where \( z_1 \in Z \cap M_1 \) and \( z_2 \in Z \cap M_2 \) clearly \( m = x + z_1 \) then \( M_1 \leq (L \cap M_1) + Z \) Similarly \( M_2 \leq (L \cap M_2) + Z \) then \( M = (L \cap M_1) + (L \cap M_2) + Z \) by modularity \( M_1 = (L \cap M_1) + [M_1 \cap (L \cap M_2) + Z] \) then \( M_1 = P_1 + [M_1 \cap (L \cap M_2) + Z] = M_1 \cap [P_1 + (L \cap M_2) + Z] \) (modular low) then \( M_1 \leq [P_1 + (L \cap M_2) + Z] \) hence \( M_1 \leq P_1 + Z \). In the same way \( M_2 \leq P_2 + Z \) then \( M = (P_1 + P_2) + Z \), \( P_1 + P_2 \) is pure (sum of two pure is pure) PSP.

**References:**


