Complex of Lascoux in Partition (4,4,4)

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Abstract
In this paper the diagrams and divided power of the place polarization \( \partial_{ij}^{(k)} \), with its capelli identities have been used, to study the complex of Lascoux in case of the partition (4,4,4).

Keywords: Divided Power, Complex of Lascoux, Capelli Identity

1. Introduction
Let \( R \) be commutative ring with 1, \( F \) be free \( R \)-module and \( D_{ij}^n \) be the divided power of degree \( n \) \([1]\). Buchsbaum in 2004 modified the boundary method \([2]\), then he and author \([3]\), studied the complex of Lascoux (characteristic zero) in the partition (2,2,2), (3,3,3) respectively, also the author in \([4]\) studied the complex of Lascoux (characteristic zero) in the partition (4,4,3) using this modified method and the technique of Letter place methods \([4]\) obtain characteristic zero.

Consider the map
\[
D_{p+k}F \otimes D_{q-k}F \rightarrow D_{p}F \otimes D_{q}F
\]
this map is the place polarization from place one to place two, where place one and place two are denoted by \( D_{p+k}F \) \( D_{q-k}F \) respectively, and the map
\[
\partial_{i2}^{(k)} : D_{p+k}F \otimes D_{q+k}F \otimes D_{r-k}F \rightarrow
\]
\[
D_{p}F \otimes D_{q}F \otimes D_{r}F
\]
is the place polarization from place two to place three \([5,6,7,8]\). In this point we can also ask for the identities in case such \( \partial_{21} \partial_{32} \) where three places have been looked, so we have to use the following equation
\[
\partial_{32}^{(l)} \circ \partial_{21}^{(l)} = \partial_{31}^{(l)}
\]
This is a typical Capelli identity \([2,3,5]\), more than
\[
\partial_{32}^{(k)} \circ \partial_{21}^{(f)} = \sum_{\alpha=0}^{\infty} \partial_{21}^{(l-\alpha)} \circ \partial_{32}^{(k-\alpha)} \circ \partial_{31}^{(\alpha)}
\]
where
\[
\partial_{ij}^{(0)} = I
\]
In general the divided power of the place polarizations satisfy the following identities in case of \( k \neq i \)
\[
\partial_{ij}^{(r)} \circ \partial_{jk}^{(s)} = \sum_{\alpha=0}^{\infty} \partial_{ij}^{(l-\alpha)} \circ \partial_{jk}^{(r-\alpha)} \circ \partial_{ik}^{(\alpha)}
\]
\[ \partial_{jk}^{(s)} \circ \partial_{ij}^{(r)} = \sum_{a \geq 0} (-1)^{a} \partial_{ij}^{(r-a)} \circ \partial_{jk}^{(r-a)} \circ \partial_{ik}^{(a)} \quad (1.4) \]

\[ \partial_{21}^{(l)} \circ \partial_{31} = \partial_{31} \circ \partial_{21}^{(k)} \quad (1.5) \]

\[ \partial_{12}^{(k)} \circ \partial_{31} = \partial_{31} \circ \partial_{12}^{(k)} \quad (1.6) \]

In this paper we study the complex of Lascoux in case of the partition \((4,4,4)\), in particular, in section two of this work we find the terms of the complex of Lascoux in case of the partition \((4,4,4)\), in section three, we study this complex as a diagram by using the divided power of the place polarization and it’s Capelli identities.

2. The Terms of Lascoux Complex in The Case of Partition \((4,4,4)\)

The terms of the Lascoux complex are obtained from the determinantal expansion of the Jacobi-Trudi matrix of the partition. The positions of the terms of the complex are determined by the length of the permutation to which they correspond \([2, 3]\).

Now in the case of the partition \(\lambda = (4,4,4)\), we have the following matrix:

\[
\begin{pmatrix}
D_4 F & D_4 F & D_6 F \\
D_3 F & D_4 F & D_5 F \\
D_2 F & D_3 F & D_4 F
\end{pmatrix}
\]

Then the Lascoux complex has the correspondence between it’s terms as follows:

\[D_4 F \otimes D_4 F \otimes D_4 F \longleftrightarrow \text{identity} \]
\[D_3 F \otimes D_3 F \otimes D_4 F \longleftrightarrow (12) \]
\[D_2 F \otimes D_3 F \otimes D_3 F \longleftrightarrow (23) \]
\[D_2 F \otimes D_2 F \otimes D_4 F \longleftrightarrow (132) \]
\[D_2 F \otimes D_2 F \otimes D_2 F \longleftrightarrow (123) \]

So, the complex of Lascoux in the case of the partition \(\lambda = (4,4,4)\) has the form:

\[D_6 F \otimes D_3 F \otimes D_3 F \]
\[D_6 F \otimes D_4 F \otimes D_2 F \rightarrow \otimes \rightarrow D_2 F \otimes D_3 F \otimes D_2 F \]
\[D_3 F \otimes D_3 F \otimes D_4 F \rightarrow \otimes \rightarrow D_4 F \otimes D_4 F \otimes D_4 F \]
\[D_4 F \otimes D_2 F \otimes D_3 F \]

3. The Complex of Lascoux as a Diagram

Consider the following diagram:

\[D_6 F \otimes D_4 F \otimes D_2 F \rightarrow D_6 F \otimes D_3 F \otimes D_3 F \rightarrow D_4 F \otimes D_4 F \otimes D_4 F \]

So, if we define

\[f_1 : D_6 F \otimes D_4 F \otimes D_2 F \rightarrow D_6 F \otimes D_3 F \otimes D_3 F \]

as

\[f_1(v) = \tilde{c}^{(1)}_{32}(v) \text{ where } v \in D_6 F \otimes D_4 F \otimes D_2 F \]

and

\[g_1(v) = \tilde{c}^{(1)}_{21}(v) \text{ where } v \in D_6 F \otimes D_4 F \otimes D_2 F \]

Now, we have to define

\[h_1 : D_6 F \otimes D_2 F \otimes D_2 F \rightarrow D_4 F \otimes D_5 F \otimes D_3 F \]

which makes the diagram A commutative i.e.

\[h_1 \circ g_1 = g_2 \circ f_1 \]

Then by using Capelli identities \((1.2)\) we have:

\[\tilde{c}^{(2)}_{21} \circ \tilde{c}^{(1)}_{32} = \tilde{c}^{(2)}_{32} \circ \tilde{c}^{(2)}_{21} - \tilde{c}^{(2)}_{21} \circ \tilde{c}^{(1)}_{31} \]

\[= \frac{1}{2} \tilde{c}^{(2)}_{32} \circ \tilde{c}^{(1)}_{21} - \tilde{c}^{(1)}_{31} \circ \tilde{c}^{(1)}_{21} \]

Thus \(h_1 = \frac{1}{2} \tilde{c}^{(1)}_{32} \circ \tilde{c}^{(1)}_{21} - \tilde{c}^{(1)}_{31} \)

On the other hand, if we define

\[h_2 : D_4 F \otimes D_4 F \otimes D_2 F \rightarrow D_4 F \otimes D_4 F \otimes D_4 F \]

as

\[h_2(v) = \tilde{c}^{(1)}_{32}(v) \text{ where } v \in D_2 F \otimes D_5 F \otimes D_3 F \]

and

\[g_3 : D_4 F \otimes D_3 F \otimes D_3 F \rightarrow D_4 F \otimes D_4 F \otimes D_4 F \]

as

\[g_3(v) = \tilde{c}^{(1)}_{21}(v) \text{ where } v \in D_2 F \otimes D_5 F \otimes D_3 F \]

Now, to make the diagram B commute, we have to define

\[f_2 : D_6 F \otimes D_3 F \otimes D_3 F \rightarrow D_5 F \otimes D_4 F \otimes D_4 F \]

such that \(g_3 \circ f_2 = h_2 \circ g_2 \) i.e.

\[\tilde{c}^{(1)}_{21} \circ f_2 = \tilde{c}^{(1)}_{32} \circ \tilde{c}^{(2)}_{21} \]

Again by using Capelli identities we get
Then \( f_2 = \frac{1}{2} \sigma_2^{(2)} \sigma_3^{(1)} \).

Now consider the following diagram

\[
\begin{array}{cccc}
D_4 F & D_3 F & D_2 F & D_1 F \\
\downarrow g_1 & \downarrow \phi & \downarrow \psi & \downarrow g_3 \\
D_4 F & D_3 F & D_2 F & D_1 F
\end{array}
\]

Define \( \phi : D_4 F \otimes D_3 F \otimes D_2 F \longrightarrow D_3 F \otimes D_2 F \otimes D_4 F \)

by \( \phi(v) = \sigma_2^{(2)}(v) \); where \( v \in D_4 F \otimes D_3 F \otimes D_2 F \)

**Proposition 3.1**

The diagram C is commutative.

**Proof:**

\[
f_2 \circ f_1 = \frac{1}{2} \sigma_2^{(2)} \sigma_3^{(1)} \circ \sigma_1^{(1)}
\]

But from (1.2) we have \( \sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_1^{(1)} \).

Then \( f_2 \circ f_1 = \sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} \).

**Proposition 3.2**

The diagram D is commutative.

**Proof:**

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

But from (1.2) we have \( \sigma_2^{(2)} \sigma_2^{(1)} = \sigma_2^{(2)} \sigma_2^{(1)} + \sigma_2^{(2)} \sigma_2^{(1)} \).

Then \( \sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_2^{(1)} \).

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
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Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

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\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

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\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]

Finally, we can define the maps \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) where

\[
\sigma_2^{(2)} \sigma_3^{(1)} = \sigma_2^{(2)} \sigma_3^{(1)} - \sigma_2^{(2)} \sigma_2^{(1)}
\]
and

\[ h_1 (c^{(1)}_{21}(x)) - g_2(x) \]

\[ = \left( \frac{1}{2} c^{(1)}_{32} \circ \partial^{(1)}_{21} - c^{(1)}_{31} \right) \circ c^{(1)}_{21}(x) - \partial^{(2)}_{21} \circ c^{(1)}_{32}(x) \]

\[ = \left( \partial^{(1)}_{32} \circ \partial^{(2)}_{21} - \partial^{(1)}_{31} \right) c^{(1)}_{21}(x) - \partial^{(2)}_{21} \circ \partial^{(1)}_{32}(x) \]

\[ = (c^{(2)}_{21} \circ c^{(1)}_{32} + c^{(2)}_{21} \circ c^{(1)}_{31}) - \partial^{(1)}_{31} - \partial^{(2)}_{21} \circ \partial^{(1)}_{32}(x) = 0 \]

So we get \((\sigma_2 \circ \sigma_3)(x) = 0\)

and

\[ (\sigma_2 \circ \sigma_3)(x_1, x_2) = \sigma_1(f_2(x_1) - \varphi(x_2), h_1(x_2) - g_2(x_1)) \]

\[ = \sigma_1(\left( \frac{1}{2} c^{(1)}_{32} \circ \partial^{(1)}_{21} + c^{(1)}_{31} \right) \circ \partial^{(1)}_{21}(x_1) - \partial^{(2)}_{32}(x_2), \]

\[ = \left( \frac{1}{2} c^{(1)}_{32} \circ \partial^{(1)}_{32} + c^{(1)}_{31} \right) \circ \partial^{(1)}_{32}(x_2) \]

\[ + c^{(1)}_{32} \left( \frac{1}{2} c^{(1)}_{32} \circ \partial^{(1)}_{21} + c^{(1)}_{31} \right) \circ \partial^{(1)}_{21}(x_1) - \partial^{(2)}_{32}(x_2) \]

\[ = (c^{(2)}_{21} \circ \partial^{(1)}_{32} + c^{(2)}_{21} \circ c^{(1)}_{31} - \partial^{(1)}_{31} \circ \partial^{(2)}_{21})(x_1) \]

But

\[ \partial^{(2)}_{21} \circ \partial^{(1)}_{32} = c^{(1)}_{32} \circ \partial^{(2)}_{21} - \partial^{(1)}_{31} \circ \partial^{(1)}_{32} \]

and

\[ \partial^{(2)}_{32} \circ \partial^{(1)}_{21} = c^{(1)}_{21} \circ \partial^{(2)}_{32} + c^{(1)}_{21} \circ \partial^{(1)}_{32} \]

which implies that \((\sigma_2 \circ \sigma_3)(x_1, x_2) = 0\).

References