Generalized Amply Cofinitely Supplemented Modules
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Abstract
Let R be an associative ring with identity. An R-module M is called generalized amply cofinitely supplemented module if every cofinite submodule of M has an ample generalized supplement in M. In this paper we proved some new results about this concept.

Keyword: Amply, cofinitely supplemented, R-module

1. Introduction
In this paper, R will denote an arbitrary ring with unity and M is a unitary left R-module.

Let M be an R-module and, recall that a submodule N of M is called small, denoted by N<<M, if N+K≠M for every proper submodule K of M (See [1], 5. 1.1). K is supplemented of N if and only if N+K=M and N∩K<<K (See [2]) where K and N are submodules of M.

M is called supplemented if every submodule of M has a supplement in M (See [2]). On the other hand, the module M is called amply supplemented if, for every submodules A, B with A+B=M, there exists a supplement C of A in M such that C⊆B (See [2]).

A submodule N of M is said to be cofinite if M/N is finitely generated (See [3]).

An R-module M is called a cofinitely supplemented module if every cofinite submodule of M has a supplement in M (See [3]). Clearly every supplemented module is cofinitely supplemented module.

An R-module M is called a cofinitely amply supplemented module if for every cofinite submodule of M has an ample supplement in M (See [4]).

Submodules A and B of an R-module M with A+B=M, B is called a generalized supplement of A in M in case A∪B⊆Rad(B), where Rad(B) is the Jacobson radical of B, (See [5])

M is called generalized supplemented module or briefly a GS-module if every submodule N of M has a generalized supplemented K in M (See[5]).

Following [5], M is called a generalized amply supplemented module or briefly a GAS-module in case M=N+K implies that N has a generalized supplement N⊆K. it is clear that every GAS-module is GS-module.

In [6] M is called generalized cofinitely supplemented if every cofinite submodule of M has a generalized supplement and denoted by GCS.

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Clearly every generalized supplemented module is generalized cofinitely supplemented.

It is shown in [6] that a generalized cofinitely supplemented module need not to be generalized supplemented, also in [8] there is one more example about such modules.

An R-module M is called generalized amply cofinitely supplemented if every cofinite submodule of M has an ample generalized supplement [7]. Clearly that every generalized amply cofinitely supplemented is generalized cofinitely supplemented, but the converse is not true [8].

2- Generalized amply cofinitely supplemented modules

An R-module M is called generalized amply cofinitely supplemented if every cofinite submodule of M has an ample generalized supplement [7].

In this section we give a new characterization for generalized amply cofinitely supplemented module when every submodule of this module is generalized cofinitely supplemented as the following lemma shows.

Theorem 2.1

If every submodule of an R-module M is generalized cofinitely supplemented, then M is generalized amply cofinitely supplemented module.

Proof:

Let N be a cofinite submodule of an R-module M. By assumption N has generalized supplement K in M such that N+K=M and N∪K⊆ Rad(K).

Now, N∪K⊆ K also N∪K is a cofinite submodule since N is a cofinite thus \( \frac{M}{N} \) is finitely generated and \( \frac{M}{N} = \frac{N+K}{N} \cong \frac{K}{N \cap K} \).

This means that \( \frac{K}{N \cap K} \) is finitely generated thus N∪K is a cofinite submodule of K, thus by assumption, there is a submodule H of K such that K= N∪K+H and (N∪K)∪H= N∪H ⊆ Rad(H). So, M=N+K=N+ N∪K+H=N+H.

Thus H is a generalized amply supplement to a cofinite submodule N of M.

An R-module is called \( \pi \)-projective module if for every two submodules A and B of M such that A+B=M, there exists a homomorphism \( f \in \text{End}(M) \) such that \( f(M) \subseteq A \) and \( (I-f)(M) \subseteq B \).[2].

In [8] proved that every weakly supplemented and \( \pi \)-projective module is amply weak supplemented module. In [9], proved that if M is cofinitely weak supplemented and \( \pi \)-projective module; then M is cofinitely amply weakly supplemented module. Also,[10] proved that every generalized supplemented module is \( \pi \)-projective module is a generalized amply supplemented module.

The following theorem is proved in [11]. Here we will give another proof with more details.

Theorem 2.2[11] Let M be a generalized cofinitely and \( \pi \)-projective module. Then M is generalized amply cofinitely supplemented module.

Proof:

Let U be a cofinite submodule of an R-module M and M=U+V where V is a submodule of M. Since M is generalized cofinitely supplemented, then U has generalized supplement X in M, i.e U+X=M and U∩X⊆ Rad(X). But M is \( \pi \)-projective, thus there exists a homomorphism \( f \in \text{End}(M) \) such that \( f(M) \subseteq V \) and \( (I-f)(M) \subseteq U \).

We claim that M=\( f(X)+U \); to prove this, let \( m \in M \Rightarrow m=u+x, u \in U, x \in X, \) hence \( m=u+x-f(x)+f(x) = u+(I-f)(x)+f(x) \in U+f(X) \).

Therefore M⊆ U+f(X). Also we have U+f(X)⊆ M thus M=U+f(X).

We claim that U∪f(X)⊆(U∪X). To prove this, let \( y \in U∪f(X) \) this implies that \( y=f(x) \), where \( x \in X \). Now, \( x-y=x-f(x) = (I-f)(x) \in U \) and since \( y \in U \), hence \( y = f(x) \in f(U∪X) \).

But U∪X⊆Rad(X), therefore \( f(U∪X) \subseteq f(Rad(X)) \), i.e. \( f(X) \subseteq f(Rad(X)) \). Also, by [1], \( f(Rad(X)) \subseteq \text{Rad}(f(X)) \). Thus U∪f(X)⊆\( f(X) \). Therefore f(X) is a generalized supplement of U in M where M=U+V.

Corollary 2.3

Every projective and generalized amply cofinitely supplemented module is generalized amply cofinitely supplemented module.

Proof:

Since every projective is \( \pi \)-projective module[2], then by theorem 2.2 we get the result.

3- Supplement submodule of a generalized amply cofinitely supplemented modules

In this section we will prove that the supplement of a generalized amply cofinitely supplemented module is amply cofinitely supplemented module.
Lemma 3.1
Every supplement submodule of a generalized amply cofinitely supplemented module is amply cofinitely supplemented modules.

Proof:
Let M be a generalized amply cofinitely supplemented module and V be any supplement submodule of M. Let V be a supplement of U in M.

Let K \subseteq V be a cofinite submodule of V such that K + T = V if we find T \subseteq such that K + T = V and K \cup T \subseteq \text{Rad}(T') then we get the result.

Since K is a cofinite supplement of V then V/K is finitely generated.

Now, \( \frac{M}{U+K} \cong \frac{U+V}{U+K} \cong \frac{V}{V \cup (U+K)} \), hence U+K is a cofinite submodule of M.

Let K+T=V, for every T \subseteq V. Since M=U+V then U+K+T=M, but M is a generally amply cofinitely supplemented module and U+K is a cofinite submodule of M, U+K has a generalized supplement submodule T' in M such that T' \subseteq T, (U+K)+ T' = M and (U+K) \cup T' \subseteq \text{Rad}(T').

Now, K+ T \subseteq V and V is a supplement of U in M, U+K+ T' = M, hence V \subseteq K+ T', thus K+ T' = V, also K \cup T \subseteq \text{Rad}(T'). Therefore K has an ample generalized supplement. ■

Corollary 3.2
Every direct summand of a generalized amply cofinitely supplemented module is generalized amply cofinitely supplemented.

Proof:
Let M be a generalized amply cofinitely supplemented module. Since every direct summand of M is a supplement in M, then by Lemma 3.1, every direct summand of M is a generalized amply cofinitely supplemented. ■

Proposition 3.3
Let M be a module, if every submodule of M is a cofinitely generalized supplemented module then M is amply generalized cofinitely supplemented.

Proof:
Let N be a cofinite submodule of M and suppose that N+K=M, where K is a submodule of M.

Now, notice that \( \frac{K}{N \cap K} \cong \frac{N+K}{N} \cong \frac{M}{N} \), since N is a cofinite submodule of M, M is finitely generated module. Thus \( \frac{K}{N \cap K} \) is finitely generated, here N \cap K is cofinite submodule of K, by assumption N \cap K has a generalized supplement H \subseteq K such that \( (N \cap K) + H = K \) and \( (N \cap K) \cap H = N \cap H \subseteq \text{Rad}(H) \), also \( N + H \geq N \cap K + H = K \) thus \( N + H \geq K + N = M \). Hence \( N + H = M \) ■

Corollary 3.4
Let R be any ring. Then the following statements are equivalent:
1- Every R-module is an amply generalized cofinitely supplemented module.
2- Every R-module is a generalized amply supplemented module. ■

References