Boundary Controllability of Nonlinear System in Quasi-Banach Spaces

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Abstract:
Sufficient conditions for boundary controllability of nonlinear system in quasi-Banach spaces are established. The results are obtained by using the strongly continuous semigroup theory and some techniques of nonlinear functional analysis, such as, fixed point theorem and quasi-Banach contraction principle theorem. Moreover, we given an example which is provided to illustrate the theory. Keywords: Boundary controllability, quasi-Banach spaces, semigroup theory, fixed point theorem.

1. Introduction:
Many scientific and engineering problems can by modeled by partial differential equations, integral equations, or coupled ordinary and partial differential equations that can be described as differential equations in infinite-dimensional spaces using semigroups. So, the study of controllability results of such problems in infinite-dimensional spaces is important. For the motivation of abstract system and the controllability of linear system, one can refer to [1,2]. Controllability of nonlinear system represented by ordinary differential equations in Banach spaces has been extensively studied by several authors. Han in [3] studied the boundary controllability of differential equations with nonlocal condition. Al-Moosawy [4] discussed the controllability and optimality of the mild solution for semilinear problems in Banach spaces, by using semigroup theory and Banach contraction principle theorem. In [5] studied the boundary controllability of integrodifferential system in Banach space. The controllability for some control problems in quasi-Banach spaces has been studied in [6] by using semigroup theory an some techniques of nonlinear functional analysis.

Since every Banach space is quasi-Banach spaces, but the converse is not true [7]. One could find a reasonable justification to accomplish the study of this paper. The purpose of this paper is to extend the study of the boundary controllability of nonlinear system in any quasi-Banach spaces by using the quasi-Banach fixed point theorem.

2. Definitions and theorems:
This section contains some definitions and theorems that will be used in the sequel.

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Definition 2.1 [8]: let \(0 < p < \infty\). Then the collection of all measurable functions \(f\) for which \(\|f\|_p\) is integrable will be denoted by \(L^p(\mu)\). For each \(f \in L^p(\mu)\), let 
\[\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}.\]

The number \(\|f\|_p\) is called the \(L^p\) norm of \(f\).

Note 2.1 [7]: The space \(L^p(\mu)\) for \(0 < p < 1\) is a vector space, but not a normed space (thus not a Banach space).

Definition 2.2 [7]: A real-valued function \(\|\|\) defined on a vector space \(V\) over a field \(F\) is called a quasi-norm if it satisfies the following properties:

1. \(\|x\| \geq 0\) \(\forall x \in V\) and \(\|x\| = 0 \iff x = 0\).
2. \(\|ax\| = |a| \|x\|\) \(\forall x \in V\), \(a \in F\).
3. \(\|x + y\| \leq \|x\| + \|y\|\) \(\forall x, y \in V\), where \(c \geq 1\) is a constant.

The pair \((V, \|\|)\) is called a quasi-normed space.

Definition 2.3 [7]: Let \((V, \|\|)\) be a quasi-normed space, then

(a) A sequence \(\{x_n\}\) in \(V\) is called convergent to the limit \(x \in V\) if, for \(\varepsilon > 0\), there exists a positive integer \(N(\varepsilon)\) such that \(\|x_n - x\| \leq \varepsilon\) \(\forall n \geq N\) (or \(\|x_n - x\| \to 0\) as \(n \to \infty\)).

(b) A sequence \(\{x_n\}\) in \(V\) is called a Cauchy sequence if, for \(\varepsilon > 0\), \(\exists N(\varepsilon) > 0\) such that \(\|x_m - x_n\| \leq \varepsilon\) \(\forall n, m \geq N\) (or \(\|x_m - x_n\| \to 0\) as \(n, m \to \infty\)).

(c) \(V\) is called a complete quasi-normed space (or quasi-Banach space) if every Cauchy sequence in \(V\) is convergent.

Definition 2.4 [7]: For \(f \in L^p(\mu)\), \(0 < p < 1\), let us define the quasi-norm of \(f\) which is denoted by \(\|f\|_q\) as follows:

\[\|f\|_q = \|f\|_p^p = \int |f|^p \, d\mu,\]

where \(\|f\|_p\) as defined in Definition 2.1.

Theorem 2.1 [7]: The space \(L^p(\mu)\) for \(0 < p < 1\), with the quasi-norm given in Definition 2.4 is a quasi-Banach space.

Definition 2.5 [9]: Let \(T\) be a mapping of a quasi-normed space \(X\) into itself, then \(T\) is called a quasi-contraction mapping if there exists a constant \(k\), \(0 \leq k < 1\) such that

\[\|T(x) - T(y)\| \leq k \|x - y\|\] \(\forall x, y \in X\).

Remark 2.1: It is clear from the above definition that every quasi-contraction mapping is uniformly continuous.

Theorem 2.2 [6]: (Quasi-Banach contraction principle)

Every quasi-contraction mapping \(T\) defined on a quasi-Banach space \(X\) into itself has a unique fixed point \(x^* \in X\). Moreover, if \(x_0\) is any point in \(X\) and the sequence \(\{x_n\}\) is defined by \(x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1})\) then

\[\lim_{n \to \infty} x_n = x^*\]

and \(\|x_n - x^*\| \leq c(k^n/1-k)^q \|x_1 - x_0\|\), where \(c \geq 1\) is a constant.

Remark 2.2 [6]:

1. Theorem 2.2 is valid for complete quasi-metric space [6] (the proof is similar).
2. Since, every closed subset \(Y\) of a quasi-Banach space \(X\) is itself a complete quasi-metric space [6].
3. Theorem 2.2 is valid for a quasi-contraction mapping defined on \(Y\) into itself.

Theorem 2.3 [6, theorem 2.2.6]: Let \(X\) and \(Y\) be a quasi-normed spaces and \(L\) be a linear transformation of \(X\) onto \(Y\). Then the inverse \(L^{-1}\) exists and is continuous on its domain of definition if and only if there exists a constant \(M > 0\), such that \(M \|y\| \leq \|L(x)\|\) for all \(x \in X\).

Definition 2.6 [9]: A family \(T(t), 0 \leq t < \infty\) of bounded linear operators on a quasi-Banach space \(X\) is called a (one-parameter) semigroup on \(X\) if it satisfies the following conditions:

1. \(T(0) = I\) (the identity operator on \(X\)).
2. \(T(t+s) = T(t)T(s)\), for each \(t, s \geq 0\).

Definition 2.7 [9]: The infinitesimal generator \(A\) of the semigroup \(T(t)\) on a quasi-Banach space \(X\) is defined by \(Ax = \lim_{t \to 0^+}(1/t)(T(t)x - x)\), where the limit exists and the domain of \(A\) is \(D(A) = \{x \in X : \lim_{t \to 0^+}(1/t)(T(t)x - x)\}\) exists.
**Definition 2.8** [9]: A semigroup \( T(t), 0 \leq t < \infty \) of bounded linear operator on a quasi-Banach space \( X \) is said to be strongly continuous semigroup (or \( C_0 \)-semigroup) if:
\[
\| T(t)x - x \| \to 0 \quad \text{as} \quad t \to 0^+ \quad \text{for all} \quad x \in X.
\]

**Theorem 2.4** [6]: Let \( X \) be a quasi-Banach space and \( T(t) \) be a \( C_0 \)-semigroup generated by \( A \). Then the following hold:

(i): For each \( x_0 \in D(A) \),
\[
T(t)x_0 \in D(A) \text{ (domain of } A \text{ ) and } A T(t)x_0 = T(t)Ax_0, \forall t \geq 0.
\]

(ii): For each \( x_0 \in D(A) \) and \( T(t)x_0 \in D(A), \)
\[
\left( \frac{d}{dt} \right)(T(t)x_0) = A T(t)x_0 = T(t)Ax_0.
\]

(iii): For each \( x_0 \in X, \)
\[
\lim_{h \to 0}(1/h) \int_0^h T(s)x_0 ds = T(t)x_0.
\]

(iv): For each \( x_0 \in X, \int_0^t T(s)x_0 ds \in D(A) \)
\[
\text{ and } A \int_0^t T(s)x_0 ds = T(t)x_0 - x_0.
\]

(v): For each \( x_0 \in D(A), \)
\[
T(t)x_0 - T(s)x_0 = \int_s^t A(T(\tau)x_0)d\tau = \int_s^t A T(\tau)x_0 d\tau.
\]

For more details about semigroup and \( C_0 \)-semigroup on a Banach space see [2, 10].

3. Controllability of Nonlinear System in Quasi-Banach Spaces:

In this section, we will study the existence theorem of the controllability of the mild solution to the nonlinear boundary-value control problem in appropriate quasi-Banach spaces, by using strongly continuous semigroup theory and quasi-Banach contraction principle theorem.

3.1. Preliminaries

Let \((E, \| \cdot \|), (U, \| \cdot \|)\) be a real quasi-Banach spaces, and \( A \) be a linear closed bounded and densely operator with \( D(A) \subseteq E, \| A \| \leq C_1 \), where \( C_1 \) is a constant and \( \tau \) be a linear operator with \( D(\tau) \subseteq E \) and range \( (\tau) \subseteq X, \) where \( X \) is a quasi-Banach space.

Consider the boundary control nonlinear system of the form
\[
\begin{align*}
\frac{dx(t)}{dt} + g(t, x(t)) &= Ax(t) +Bu(t) + F(t, N(t, x(t))) \\
\tau \{ x(t) + g(t, x(t)) \} &= \varepsilon u(t), t \in J = [0, b]
\end{align*}
\]

(1)

Where \( B_1 : U \to X \) is a linear continuous operator, the control function \( u(\cdot) \in U \) a quasi-Banach space of admissible control functions. Let \( A_1 : E \to E \) be the linear operator defined by:
\[
A_1 x = Ax, x \in D(A_1), \text{where } D(A_1) = \{ x \in D(A): \tau x = 0 \}.
\]

Let \( B_r = \{ x \in E: \| x \| \leq r \}, \) for some \( r > 0 \).

We shall make the following hypotheses:

(i): \( D(A) \subseteq D(\tau) \) and the restriction of \( \tau \) to \( D(A) \) is continuous relative to graph norm of \( D(A_1) \).

(ii): The operator \( A_1 \) is the infinitesimal generator of a \( C_0 \)-semigroup \( T(t) \) and there exist a constant \( M > 0 \), such that \( \| T(t) \| \leq M \).

(iii): There exist a linear continuous operator \( B_2 : U \to E \) such that \( AB_2 \in L_p(U, E) \) where \( 0 < p < 1, \tau (B_2 U) = B_1 u \) for all \( u \in U, \) also \( B_2 u(t) \) is continuously differentiable.

(iv): For all \( t \in (0, b) \) and \( u \in U, \text{ and } T(t) B_2 u \in D(A_1) \), moreover there exist a positive function \( v_0 \in L^1(0, b) \), such that \( q_{A_1} T(t) B_2 u \| \leq v_0(t) \) almost everywhere \( t \in (0, b) \).

(v): The nonlinear operator \( N : J \times E \to E \) is continuous and satisfies Lipschitz condition on the second argument, such that for all \( x_1, x_2 \in B_r \) and a positive constant \( M_3 \), we have:
\[
q_{N(t, x_1)} - N(t, x_2) \| \leq M_3 q_{x_1 - x_2} .
\]

(vi): The nonlinear operator \( F : J \times E \to E \) is continuous and satisfies Lipschitz condition on the second argument, for \( x_1, x_2 \in B_r \) and the positive constants \( M_1 \) and \( M_2 \), we have:
\[
q_{F(t, N(t, x_1))} - F(t, N(t, x_2)) \| \leq M_1 q_{x_1 - x_2} .
\]

and \( M_2 = \max_{t \in J} q_{F(t, N(t, 0))} \).

(vii): \( B : U \to E \) is bounded linear operator, \( q_{B} \leq C \), where \( C \) is a positive constant.

(viii): The nonlinear operator \( g : J \times E \to E \), satisfy Lipschitz condition on the second argument, let \( L_1, L_2 > 0 \) be constants, such that for all \( x_1, x_2 \in B_r \) we have:
\[
q_{g(t, x_1)} - g(t, x_2) \| \leq L_1 q_{x_1 - x_2} \] and \( L_2 = \max_{t \in J} q_{g(t, 0)} \).

The main aim of this section is to find the mild solution of (1).
Now let \( x(t) \) be the solution of (1). Then we can define a function
\[
Z(t) = x(t) + g(t, x(t)) - B_2 u(t) 
\]
(2)
From our assumption we want to show that \( Z(t) \in D(A_1) \), by (2):
\[
\tau(Z(t)) = \tau(x(t) + g(t, x(t)) - B_2 u(t)) 
\]
\[
= \tau(x(t) + g(t, x(t)) - \tau(B_2 u(t))) 
\]
By condition (iii) and equation (1) we have \( \tau(Z(t)) = B_1 u(t) - B_1 u(t) = 0 \)
So, by definition of \( D(A_1) \), we have \( Z(t) \in D(A_1) \), and \( A Z(t) = A_1 Z(t) \).
Therefore, from (2), (1) can be written in term \( A_1 \) and \( B_2 \) as :
\[
\frac{d}{dt}\left(x(t) + g(t, x(t))\right) = A_1 Z(t) - g(t, x(t)) + B_2 u(t) + F(t, N(t, x(t))) 
\]
\[
= A_1 Z(t) - A g(t, x(t)) + B_2 u(t) + F(t, N(t, x(t))) 
\]
(3)
By differentiable, if \( x \) is continuously differentiable on \([0, b]\), then \( Z(t) \) can be defined as a mild solution to the Cauchy problem:
\[
\frac{d}{dt} Z(t) = g(t, x(t)) - B_2 \frac{d}{dt} u(t), 
\]
by equation (3), we get :
\[
\frac{d}{dt} Z(t) = A_1 Z(t) - Ag(t, x(t)) + B_2 u(t) + F(t, N(t, x(t))) 
\]
(4)
Since in condition (ii), we have \( T(t) \), \( \forall t \geq 0 \) is a \( C_0 \)-semigroup generated by the linear operator
\( A_1 \) and \( Z(t) \) is a solution of (4), then by theorem 2.4, the function \( H(s) = T(t-s)Z(s) \) is differentiable for \( 0 < s < t \), and
\[
\frac{d}{ds} H(s) = T(t-s) \frac{d}{ds} Z(s) + Z(s) \frac{d}{ds} T(t-s), 
\]
thus by (4) and theorem 2.4 (ii), we have :
\[
\frac{d}{ds} H(s) = T(t-s)[A_1 Z(s) - A g(s, x(s)) + AB_2 u(s) + Bu(s) - B_2 \frac{d}{ds} u(s) + 
F(s, N(s, x(s))) + Z(s)[-A_1 T(t-s)] 
= T(t-s)A_1 Z(s) - T(t-s)A g(s, x(s)) + 
+T(t-s)AB_2 u(s) + 
T(t-s)Bu(s) - T(t-s)B_2 \frac{d}{ds} u(s) 
+T(t-s)F(s, N(s, x(s))) 
- T(t-s)A_1 Z(s) \)
and by integration from 0 to \( t \), yield.
\[
H(t) - H(0) = -\int_0^t T(t-s)A g(s, x(s)) ds + 
\int_0^t T(t-s)AB_2 u(s) ds + 
\int_0^t T(t-s)Bu(s) ds - 
\int_0^t T(t-s)B_2 \frac{d}{ds} u(s) ds + \int_0^t T(t-s)[T(t-s)A_1 Z(s) - T(t-s)A g(s, x(s)) + 
+T(t-s)AB_2 u(s) + 
T(t-s)Bu(s) - T(t-s)B_2 \frac{d}{ds} u(s) 
+T(t-s)F(s, N(s, x(s))) ds 
- T(t-s)A_1 Z(s)] ds 
\]
By integrate the part \( \int_0^t T(t-s)B_2 \frac{d}{ds} u(s) ds \) in (5) by part, we get :
\[
x(t) = T(t)x_0 + \int_0^t T(t-s)g(s, x_0) - g(s, x(t)) ds + 
\int_0^t T(t-s)A g(s, x(s)) ds + 
\int_0^t T(t-s)Bu(s) ds - 
\int_0^t T(t-s)B_2 \frac{d}{ds} u(s) ds + \int_0^t ([T(t-s)A_1 Z(s) - T(t-s)A g(s, x(s)) + 
+T(t-s)AB_2 u(s) + 
T(t-s)Bu(s) - T(t-s)B_2 \frac{d}{ds} u(s)] ds 
- T(t-s)A_1 Z(s)] ds 
\]
(6)
Defintion 3.1: A function \( x : [0, b] \rightarrow E \) defined by the Integro equation (6) is called a mild solution of (1) if \( x \) is continuously differentiable on \([0, b]\), continuous on \([0, b]\) and \( x(t) \in E \) for \( 0 < s < t \).

Defintion 3.2: The boundary value control problem (1) is said to be controllable on the interval \( J = [0, b] \) if for every \( x_0, x_1 \in E \), there exists a control \( u \in U \) such that the mild solution (6) satisfied \( x(b) = x_1 \).

Here we further consider the following additional conditions :
\( (ix) \): There exists a constant \( k > 0 \) such that
\[
\int_0^b v_0(i) dt \leq k_1. 
\]
(x): Define the linear continuous operator \( \mathcal{W} \) from \( U \) onto \( E \) as follows
\[
\mathcal{W}u = \int_0^b [T(b-s)A - A_1 T(b-s)]B_2 + T(b-s)\mathcal{B} u(s)ds
\]
and suppose that for every \( u \in \mathcal{U} \), there exists a constant \( k > 0 \) such that
\[
\|k_u\| \leq \mathcal{W}u.
\]
From the above condition (x) and theorem 2.3, we see that the inverse operator \( \mathcal{W}^{-1} \) exists and is continuous (bounded), i.e., the operator \( \mathcal{W}^{-1} \):
\[
\text{Rang } \mathcal{W} \to \mathcal{U} \text{ defined by } \mathcal{W}^{-1}(\mathcal{W}u(t)) = u(t)
\]
exists and there exists a positive constant \( k_2 > 0 \) such that \( \|\mathcal{W}^{-1}\| \leq k_2 \).

(xi):
\[
C_2 M_\varphi \|x_0\| + C_2 M_2 h_1 + C_2 C_4 h_2 + C_2 bM_1 C_4 k_2 + C_2 b M_2 M_3 h_3 + (C_2 bM_1 M_2)\|
\]
\[
+ \mathcal{B}_C \{L_n \|x_1\| + (L_2 + C_4 h_2) + bM_1 C_4 k_2 + bM_1 h_2 \}
\]
where \( C_i \geq 1, i = 2,3,4,5,6 \), are constants and
\[
h_1 = L_1, h_2 = L_2, k_2 = L_1 + L_2 \quad \text{and} \quad h_3 = M_1 + M_2.
\]

(xii):
\[
\mathcal{W} = C_7 L_1 + C_7 M_4 L_1 + C_7 b M M_1 +
\]
\[
C_7 (b M C_1 + k_2 h_2 + b M C_2)
\]
\[
(b M C_1 L_2 + b M M_2), \quad \text{such that} \quad 0 \leq q < 1 \quad \text{where} \quad \mathcal{C} \geq 1 \text{ be a constant.}
\]

3.2. Main Result
Theorem 3.1: If the hypotheses (i)-(xii) are satisfied, then the boundary control nonlinear system (1) is controllable on \( J \).

Proof. By definition 3.2, and condition (x) we have
\[
x_1 = x(b) = T(b)x_0 + T(b)g(0,x_0) - g(b,x) - \int_0^b T(b-s)A g(s,x(s)) ds +
\]
\[
\int_0^b T(b-s)F(s,N(s,x(s))) ds + \mathcal{W}u(t).
\]
Since \( u(t) = \mathcal{W}^{-1}(\mathcal{W}u(t)) \), then we get that
\[
u(t) = \mathcal{W}^{-1}[x_1 - T(b)x_0 - T(b)g(0,x_0) + g(b,x_1) + \int_0^b T(b-s)A g(s,x(s)) ds -
\]
\[
\int_0^b T(b-s)F(s,N(s,x(s))) ds] \quad \text{and} \quad (7)
\]
Let \( Y = C(J,B_r) \). Using this control (7), we shall show that the operator \( \mathcal{O} \) defined by:
\[
\mathcal{O}x(t) = \mathcal{T}(t)x_0 + \mathcal{T}(t)g(0,x_0) - \mathcal{G}(t,x(t)) - \int_0^b \mathcal{T}(b-s)A g(s,x(s)) ds +
\]
\[
\int_0^b \mathcal{T}(b-s)F(s,N(s,x(s))) ds + \int_0^b \mathcal{T}(t-s)A_1 (t-s)B_2 +
\]
\[
T(t-s)B \mathcal{W}^{-1}[x_1 - T(b)x_0 - T(b)g(0,x_0) + g(b,x_1) - \int_0^b T(b-s)A g(s,x(s)) ds -
\]
\[
\int_0^b T(b-s)F(s,N(s,x(s))) ds] \quad \text{has a unique fixed point.}
\]
First we show that \( \mathcal{O} \) map \( Y \) into itself, for \( x \in Y \) to show that \( \mathcal{O}x(t) \| \leq r. \)
\[
\|\mathcal{O}x(t)\| \leq C_2 \mathcal{T}(x_0)\| + \|\mathcal{T}(t)g(0,x_0)\| + \|\mathcal{G}(t,x(t))\| +
\]
\[
\int_0^b \|\mathcal{T}(b-s)\| \|\mathcal{T}(t-s)\| \mathcal{A}\|\|\mathcal{G}(s,x(s))\| ds +
\]
\[
\left\{ \int_0^b \|\mathcal{T}(t-s)\| \mathcal{A}\|\|\mathcal{G}(s,N(s,x(s)))\| ds + \right\}
\]
\[
+ \|\mathcal{T}(t-s)B\|\|\mathcal{W}^{-1}\| \|\mathcal{W}^{-1}\| + \|\mathcal{T}(b)\|\|\mathcal{G}(0,x_0)\| + \|\mathcal{G}(b,x_1)\| +
\]
\[
\left\{ \int_0^b \|\mathcal{T}(b-s)\| \mathcal{A}\|\|\mathcal{G}(s,x(s))\| ds + \right\}
\]
\[
\left\{ \int_0^b \|\mathcal{T}(t-s)\| \mathcal{A}\|\|\mathcal{G}(s,N(s,x(s)))\| ds + \right\}
\]
\[
\left\{ \int_0^b \|\mathcal{T}(b-s)\| \mathcal{A}\|\|\mathcal{G}(s,x(s))\| ds + \right\},
\]
where \( C_2 \geq 1 \) be a constant. By (ii),(iv), and
\[
\|\mathcal{A}\| \leq C_1, \quad \text{we} \quad \text{have}
\]
\[
\|\mathcal{O}x(t)\| \leq C_2 M_\varphi \|x_0\| +
\]
\[
C_2 M_\varphi \|g(0,x_0)\| + g(0,0) + g(0,0) +
\]
\[
C_2 \mathcal{A}\|\|\mathcal{G}(t,x(t)) - \mathcal{G}(t,0) + \mathcal{G}(t,0)\|\|\mathcal{A}\|\|\mathcal{G}(t,x(t)) - \mathcal{G}(t,0) + \mathcal{G}(t,0)\|
\]
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Since \( x \in B_n \), then \( q \|kx\| \leq r \). Thus by conditions (vi), (vii) and (viii) we get that
\[
q \|\partial x(t)\| \leq C_2 M q \|x_0\| + C_2 M C_3(L_1 q \|x_0\| + L_2) + C_2 M C_3(L_1 q \|x_0\| + L_2) + C_2 M C_3(M_1 r + M_2) + (C_2 M q \|A B_2\| + b M C + k_1) k_2 \|x_1\| + M q \|x_0\| + M C_3(L_1 q \|x_0\| + L_2) + C_2 M C_3(L_1 q \|x_0\| + L_2) + b M C_3(M_1 r + M_2)\],

where \( C_i \geq 1, i = 3,4,5,6 \), are constants. By condition (xi) we have
\[
q \|\partial x(t)\| \leq C_2 M q \|x_0\| + C_2 M C_3(L_1 q \|x_0\| + L_2) + C_2 M C_3(M_1 r + M_2)\],

Thus \( \partial \) map \( Y \) into itself. Now for \( x_1, x_2 \in Y \) we have
\[
q \|\partial x_1(t) - \partial x_2(t)\| \leq C_i q \|g(t, x_1(t)) - g(t, x_2(t))\| + \int_0^r \|T(t - s)\| q \|A\| ds + \int_0^r \|T(t - s)\| q \|F(s, N(s, x(s))) - F(s, N(s, x(s)))\| ds + \int_0^r \|T(t - s)\| q \|F(s, N(s, x(s))) - F(s, N(s, x(s)))\| ds + \int_0^r \|T(t - s)\| q \|A\| ds\]
and by condition (xii) we see that
\[
q \|\partial x_1(t) - \partial x_2(t)\| \leq C_7 L_1 q \|x_1(t) - x_2(t)\| + C_7 M q \|x_1(t) - x_2(t)\| + C_7 M q \|x_1(t) - x_2(t)\| + C_7 M q \|x_1(t) - x_2(t)\| + C_7 M q \|x_1(t) - x_2(t)\| + C_7 M q \|x_1(t) - x_2(t)\|\]

Thus \( \partial \) is a contraction mapping. Hence by quasi-Banach contraction principle theorem (Theorem 2.2) there exist a unique fixed point \( x(t) \in Y \) such that \( \partial x(t) = x(t) \). Any fixed point is a mild solution of (1) on \( J \) which satisfies \( x(b) = x_1 \). Thus the system (1) is controllable on.

### 3.3 Application

The Leslie model [7] is a powerful tool used the matrices to determine the growth of a population as well as the age distribution within a population over certain time interval.

**Definition 3.3** [7]: An infinite matrix \((a_{ij})\) whose elements satisfy
\[
a_{ij} = \begin{cases} F_i & i = 1 \text{ and } j = 1, \ldots, \text{otherwise} \end{cases}
\]

is the average reproduction of females in the \( i \)-th age class, and \( 0 < F_i < 1 \) is the survival rate of a females in the \( i \)-th age class, is called an infinite dimensional Leslie matrix.

Let \((a_{ij})\) be Leslie matrix whose elements are function in the quasi-Banach space \( L_p \) for \( 0 < p < 1 \).

**Theorem 3.2** [7]: An infinite-dimensional Leslie matrix \((a_{ij})\) defines a bounded linear operator from \( L_p \) into \( L_p \) when \( 0 < p < 1 \).
Theorem 3.3 [7] : An infinite- dimensional Leslie matrix \((\alpha_{ij})^n_{i,j}\) defines a compact linear operator from \(L_p\) into \(L_p\) when \(0 < p < 1\).

Now, let \(E = U = X = L_p\) for \(0 < p < 1\) be real quasi-Banach spaces, and consider the problem (1), where \(A = (\alpha_{ij})^n_{i,j}\) is an infinite-dimensional Leslie matrix, \(B\) is a matrix whose elements are functions in the quasi-Banach space \(L_p\), \(0 < p < 1\), \(B_1\) is the identity operator, and assume that the operators \(g(\ldots)\) and \(F(\ldots)\) in (1) are identical to zero operator.

Then by theorems 3.2 and 3.3, the matrix \(A = (\alpha_{ij})^n_{i,j}\) defines a bounded (compact) linear operator from \(L_p\) into \(L_p\) for \(0 < p < 1\).

By the same way we see that the operator \(B\) is bounded.

Thus the operator \(A\) is the infinitesimal generator of a \(C_0\)-semigroup defined by \(T(t) = e^{At} = \sum_{k=0}^{\infty} \left(t^k A^k/k!\right), t \geq 0\), which is bounded [10].

Therefore it is not difficult to check that all assumptions of theorem 3.1, are satisfied for the above problem [9].

4. Conclusions:
In this paper we extend the study of controllability of control problem in any quasi-Banach spaces. Thus the concepts of a quasi-Banach space are introduced, such as a quasi-Banach contraction principle theorem, strongly continuous semigroup and used it to prove theorem deals with the controllability for nonlinear boundary-value control problem in the quasi-Banach spaces.

5. Future work:
The observability and optimality for the problem (1) may be considered.

References