Fuzzy Soft Modules Over Fuzzy Soft Rings

Amjad Hamead Alhusiny
Department of computer science, College of education, University of Kufa, Alnajaf, Iraq

Abstract.
Let $M$ be an $R$-module, and let $A \neq \emptyset$ be a set, let $(F,A)$ be a soft set over $M$. Then $(F,A)$ is said to be a fuzzy soft module over $M$ iff $\forall a \in A, F(a)$ is a fuzzy submodule of $M$. In this paper, we introduce the concept of fuzzy soft modules over fuzzy soft rings and some of its properties and we define the concepts of quotient module, product and coproduct operations in the category of FSFS modules.

Keywords: Fuzzy Soft Modules, Fuzzy Soft Rings, quotient module

1. Introduction
Most of our traditional mathematical tools are deterministic and precise in character. In the other hand many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. Molodtsov [1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. Later other authors Maji et al. [2-4] have further studied the theory of soft sets and also introduced the concept of fuzzy soft set, which is a combination of fuzzy set [5] and soft set. Thereafter, Aktas and Cagman [6] have introduced the notion of soft groups. Aygunoglu and Aygun [7] have generalized the concept of Aktas and Cagman [6] and introduce fuzzy soft group. F. Feng et al. [8] gave soft semirings and U. Acar et al. [9] introduced initial concepts of...
soft rings. Ghosh et al. [10] gave the notion of fuzzy soft rings. The definition of fuzzy modules is given by some authors [11-13] Qiu-Mei Sun et al. [14] defined soft modules and investigated their basic properties. Gunduz and Bayramov [15] introduced a basic version of fuzzy soft module theory. In this paper the main purpose is to introduce a basic version of fuzzy soft module over fuzzy soft ring, which extends the notion of module by including some algebraic structures in soft sets. And finally some basic properties of fuzzy soft module over fuzzy soft ring has been investigated.

2. Preliminaries:
Definition (2.1) [1] Let $U$ be an initial universe set and $E$ be the set of parameters. Let $P(U)$ denotes the power set of $U$. A pair $(F,E)$ is called a soft set over $U$, where $F$ is a mapping given by $F:E \rightarrow P(U)$.

Definition (2.2) [2] Let $U$ be an initial universe set and $E$ be the set of parameters. Let $A \subseteq E$. A pair $(F,A)$ is called fuzzy soft set over $U$, where $F$ is a mapping given by $F:A \rightarrow I^U$, where $I^U$ denotes the collection of all fuzzy subsets of $U$.

Definition (2.3) [16] A binary operation $\ast: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous $t$-norm if it satisfies the following conditions:
(i) $\ast$ is commutative and associative,
(ii) $\ast$ is continuous,
(iii) $a \ast 1 = a$ for all $a \in [0,1]$,
(iv) $a \ast b \leq c \ast d$ whenever $a \leq c, b \leq d$, and $a,b,c,d \in [0,1]$.

Definition (2.4) [16] A binary operation $\circ: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous $t$-conorm if it satisfies the following conditions:
(i) $\circ$ is commutative and associative,
(ii) $\circ$ is continuous,
(iii) $a \circ 0 = a$ for all $a \in [0,1]$,
(iv) $a \circ b \leq c \circ d$ whenever $a \leq c, b \leq d$ and $a,b,c,d \in [0,1]$.

Definition (2.5) [6] Let $X$ be a group and $(F,A)$ be a soft set over $X$. Then $(F,A)$ is said to be a soft group over $X$ if and only if $F(a)$ is a subgroup of $X$ for each $a \in A$.

Definition (2.6) [7] Let $X$ be a group and $(F,A)$ be a fuzzy soft set over $X$. Then $(F,A)$ is said to be a fuzzy soft group over $X$ if and only if for each $a \in A$ and $x,y \in X$,
(i) $F_a(x \cdot y) \geq F_a(x) \ast F_a(y)$
(ii) $F_a(x^{-1}) \geq F_a(x)$
Where $F_a$ is the fuzzy subset of $X$ corresponding to the parameter $a \in A$.

Remark (2.7) $F_a(x^{-1}) = F_a(x)$. Proof: Let $x,y \in X$, such that $x^{-1} = y$, which means that $y^{-1} = x \cdot y$, since $F_a(x^{-1}) \geq F_a(x)$ then $F_a(y^{-1}) \geq F_a(y)$, and so $F_a(x) \geq F_a(y) = F_a(x^{-1})$, and thus $F_a(x^{-1}) = F_a(x)$.

Definition (2.8) [10] Let $f$ and $g$ be any two fuzzy subset of a ring $R$. Then $f \circ g$ is a fuzzy subset of $R$ defined by 
\[
(f \circ g)(z) = \begin{cases} 
\sup_{x,y \in z} \min\{f(x), g(y)\}, & \text{if } z \text{ is expressed as } z = x \cdot y, \text{ where } x,y \in R. \\
0, & \text{otherwise}
\end{cases}
\]

Definition (2.9) [10] The intersection of two fuzzy soft sets $(F,A)$ and $(G,B)$ over the same universe $U$ is denoted by $(F,A) \cap (G,B)$ and defined by a fuzzy soft set $(H,C)$ where $C = A \cap B$ and $C: C \rightarrow [0,1]^U$ such that for each $e \in C$,
$$H_e(x) = \{ (x, H_e(x)) : x \in U \}$$
Where $H_e(x), F_e(x), G_e(x)$ are the fuzzy subset of $U$ corresponding to the parameter $e \in C$.

Definition (2.10) [10] The union of two fuzzy soft sets $(F,A)$ and $(G,B)$ over the same universe $U$ is denoted by $(F,A) \cup (G,B)$ and defined by a fuzzy soft set $(H,C)$ where
\[ C = A \cup B \text{ and } \mathcal{H}: C \to [0,1]^U \text{such that for each } e \in C, \]
\[ \mathcal{H}(e) = \{(x,F_e(x)) : x \in U\}, \text{ if } e \in A - B = \{(x,G_e(x)) : x \in U\}, \text{ if } e \in B - A = \{(x,H_e(x)) : x \in U\}, \text{ if } e \in A \cap B. \]

Where \( H_e(x), F_e(x), G_e(x) \) are the fuzzy subset of \( U \) corresponding to the parameter \( e \in C \).

**Definition (2.11)** [4] If \((f,A)\) and \((g,B)\) are two soft sets, then \((f,A) \wedge (g,B)\) is denoted as \((h,A \times B)\), where \(h(a,b) = h_{ab} = f_a \wedge g_b\), \(\forall (a,b) \in A \times B\).

**Definition (2.12)** [10] Let \((R,+,\cdot)\) be a ring and \(E\) be a parameter set and \(A \subseteq E\). Let \(\mathcal{R}\) be a mapping given by \(\mathcal{R} : A \to P(R)\). Then \(\mathcal{R}(A)\) is called a soft ring over \(R\) if and only if for each \(a \in A, \mathcal{R}(a)\) is a subring of \(R\) i.e.

\(i\) \(x, y \in \mathcal{R}(a) \Rightarrow x + y \in \mathcal{R}(a)\),
\(ii\) \(x \in \mathcal{R}(a) \Rightarrow -x \in \mathcal{R}(a)\),
\(iii\) \(x, y \in \mathcal{R}(a) \Rightarrow x \cdot y \in \mathcal{R}(a)\).

**Definition (2.13)** [10] Let \((R,+,\cdot)\) be a ring and \(E\) be a parameter set and \(A \subseteq E\). Let \(\mathcal{R}\) be a mapping given by \(\mathcal{R} : A \to [0,1]^R\), where \([0,1]^R\) denotes the collection of all fuzzy subsets of \(R\). Then \(\mathcal{R}(A)\) is called a fuzzy soft ring over \(R\) if and only if for each \(a \in A, \mathcal{R}(a)\) the corresponding fuzzy subset \(\mathcal{R}_a\) of \(R\) is a fuzzy subring of \(R\) i.e.

\(\forall x, y \in R,\)
\(i\) \(\mathcal{R}_a(x + y) \geq \mathcal{R}_a(x) \ast \mathcal{R}_a(y)\)
\(ii\) \(\mathcal{R}_a(-x) \geq \mathcal{R}_a(x)\)
\(iii\) \(\mathcal{R}_a(x \cdot y) \geq \mathcal{R}_a(x) \ast \mathcal{R}_a(y)\).

**Theorem (2.14)** [10] Let \((\mathcal{R},A)\) be a fuzzy soft set over \(R\), then \((\mathcal{R},A)\) is a fuzzy soft ring over \(R\) if and only if for each \(a \in A, x, y \in R\) the following conditions hold:

\(i\) \(\mathcal{R}_a(x - y) \geq \mathcal{R}_a(x) \ast \mathcal{R}_a(y)\)
\(ii\) \(\mathcal{R}_a(x \cdot y) \geq \mathcal{R}_a(x) \ast \mathcal{R}_a(y)\).

**Definition (2.15)** [14] Let \((F,A)\) be a soft set over \(M\). \((F,A)\) is said to be a soft module over \(M\) if and only if \(F(x) < M\) for all \(x \in A\).

**Definition (2.16)** [14] Let \((F,A)\) and \((G,B)\) be two soft modules over \(M\) and \(N\) respectively. Then \((F,A) \times (G,B) = (H,A \times B)\) is defined as \(H(x,y) = F(x) \times G(y)\) for all \(x,y \in A \times B\).

**Proposition (2.17)** [14] Let \((F,A)\) and \((G,B)\) be two soft modules over \(M\) and \(N\) respectively. Then \((F,A) \times (G,B)\) is soft module over \(M \times N\).

**Definition (2.18)** [14] Let \((F,A)\) and \((G,B)\) be two soft modules over \(M\) and \(N\) respectively, \(f : M \to N, g : A \to B\) be two functions. Then we say that \((f,g)\) is a soft homomorphism if the following conditions are satisfied:

\(1\) \(f\) is a homomorphism from \(M\) onto \(N\),
\(2\) \(g\) is a mapping from \(A\) onto \(B\), and
\(3\) \(f(F(x)) = G(g(x))\) for all \(x \in A\).

**Definition (2.19)** [17] Let \((F,A)\) be a fuzzy soft set over \(G\). Then \((F,A)\) is said to be a fuzzy soft group over \(G\) if and only if \(F(x)\) is a fuzzy subgroup of \(G\), for all \(x \in A\).

**Definition (2.20)** [15] Let \((F,A)\) be a fuzzy soft set over \(M\). Then \((F,A)\) is said to be a fuzzy soft module over \(M\) if and only if \(\forall a \in A, F(a)\) is a fuzzy submodule of \(M\) and denoted as \(F_a\).

**Definition (2.21)** [15] Let \((F,A)\) and \((H,B)\) be two fuzzy soft modules over \(M\). Then \((F,A)\) is called a fuzzy soft submodule of \((H,B)\) if

\(i\) \(A \subseteq B\)
\(ii\) For all \(a \in A, F_a\) is a fuzzy submodule of \(H_a\).

**Definition (2.22)** [15] Let \((F,A)\) and \((H,B)\) be two fuzzy soft modules over \(M\) and \(N\) respectively, and let \(f : M \to N\) be a homomorphism of modules, and let \(g : A \to B\) be a mapping of sets. Then we say that
\((f, g): (F, A) \rightarrow (H, B)\) is a fuzzy soft homomorphism of fuzzy soft modules, if the following condition is satisfied:

\[ f(F_a) = H(g(a)) = H_g(a) \cdot \]

We say that \((F, A)\) is a fuzzy soft homomorphic to \((H, B)\).

Note that for \(\forall a \in A, f: (M, F_a) \rightarrow (N, H_g(a))\) is a fuzzy homomorphism of fuzzy modules.

**Theorem (2.23) [15]** Let \((F, A)\) and \((H, B)\) be two fuzzy soft modules over \(M\). Then their intersection \((F, A) \cap (H, B)\) is a fuzzy soft module over \(M\).

**Theorem (2.24) [15]** Let \((F, A)\) and \((H, B)\) be two fuzzy soft modules over \(M\). Then \((F, A) \wedge (H, B)\) is a fuzzy soft module over \(M\).

**Definition (2.25) [15]** \(\prod_{i \in I} (F_i, A_i)\) is said to be direct product of soft modules.

**Proposition (2.26) [15]** Let \(\{ (F_i, A_i) \}_{i \in I}\), be a family of soft modules over \(\{ M_i \}_{i \in I}\) and \(\{ (H_i, B_i) \}_{i \in I}\), be a family of soft modules over \(\{ N_i \}_{i \in I}\) and \((f_i, g_i):(F_i, A_i) \rightarrow (H_i, B_i)\) be a soft homomorphism of soft modules for each \(i \in I\). Then

\[ \left( \prod_{i \in I} F_i, \prod_{i \in I} g_i \right): \prod_{i \in I} (F_i, A_i) \rightarrow \prod_{i \in I} (H_i, B_i) \]

is a soft homomorphism of soft modules.

**Proposition (2.27) [15]** \(\Pi: \Pi FSM \rightarrow FSM\) is a functor.

Now, let parameter set of \(\{ (F_i, A_i) \}_{i \in I}\) be fixed point. We denote fixed point of \(\alpha_i\) as \(a_{0i}\) and let \(F_i(a_{0i}) = 0\). For \(A = \prod_{i \in I} A_i\) and \(M = \oplus_{i \in I} M_i\), we define the mapping \(F: A \rightarrow M\) by \(F(\alpha) = \oplus_{i \in I} F_i(\alpha_i)\) for all \(\alpha = \{\alpha_i\} \in A\). Then, \((F, A)\) is a soft module over \(M\).

**Definition (2.28) [15]** \((F, A)\) is said to be direct sum of \(\{ (F_i, A_i) \}_{i \in I}\), and denoted as \( \oplus_{i \in I} (F_i, A_i)\).

The mapping \(\varphi: A \rightarrow \prod_{i \in I} A_i\) is defined by \(\varphi(\alpha) = \{\alpha_i\} = \{\alpha_{0i}\} \text{ if } i \neq j\) and \(\alpha\) if \(i = j\).

Also for embedding mapping \(q_j: M_j \rightarrow \oplus_{i \in I} M_i\),

\(q_j(\varphi_j):(F_j, A_j) \rightarrow (F, A)\)

is a soft homomorphism of soft modules.

**Proposition (2.29) [15]** Let \(\{ (F_i, A_i) \}_{i \in I}\) and \(\{ (H_i, B_i) \}_{i \in I}\) be family of soft modules over \(\{ M_i \}_{i \in I}\) and \(\{ N_i \}_{i \in I}\), respectively, and let \((f_i, g_i):(F_i, A_i) \rightarrow (H_i, B_i)\) be a soft homomorphism of soft modules. Then

\[ \left( \oplus_{i \in I} F_i, \prod_{i \in I} g_i \right): \oplus_{i \in I} (F_i, A_i) \rightarrow \oplus_{i \in I} (H_i, B_i) \]

is a soft homomorphism of soft modules.

**Proposition (2.30) [15]** \(\oplus: \Pi FSM \rightarrow FSM\) is a functor.

Let \(M\) and \(N\) be, respectively, right and left modules over \(R\) (ring). Let \((F_i, A_i)\) and \((G_i, B_i)\) be two soft modules over \(M\) and \(N\), respectively. We consider tensor product of modules as \(M \otimes N\). The mapping \(F \otimes G: A \times B \rightarrow M \otimes N\) is defined by \((F \otimes G)(a, b) = F(a) \otimes G(b)\) for \(\forall (a, b) \in A \times B\).

**Proposition (2.31) [15]** \((F \otimes G, A \times B)\) is a soft module over \(M \otimes N\).

**Definition (2.32) [15]** \((F \otimes G, A \times B)\) is said to be tensor product of \((F, A)\) and \((G, B)\) and denoted by \((F, A) \otimes (G, B)\).

3. Fuzzy soft modules over fuzzy soft rings:

**Definition (3.1):** Let \(M\) be an \(R\) - module, \((F, A)\) be a fuzzy soft module over \(M\), and \((\mathcal{R}, A)\) be a fuzzy soft ring over \(R\), then \((F, \mathcal{R}, A)\) is called fuzzy soft module over fuzzy soft ring if and only if \((F, A)\) is an \((\mathcal{R}, A)\) - module.

**Remark (3.2):** 1) We shall denote the category of fuzzy soft modules over fuzzy soft rings by \(FSFS\) modules.

2) For convenience we denote the fuzzy soft module over fuzzy soft ring \((F, \mathcal{R}, A)\) by \((F, A)\), wherever there is no risk of confusion.

**Example (3.3):** Let \(M_{n \times n}(\mathbb{R})\) be the set of all \(n \times n\) matrices over \(\mathbb{R}\), and...
$R = A = M_{n \times n}(R)$, define the function $\mathcal{R}: A \rightarrow [0, 1]^R$ by $\mathcal{R}(B) = \{C \cdot B \mid C \in M_{n \times n}(R)\}$ for all $B \in A$. Then $(\mathcal{R}, A)$ is a fuzzy soft ring over $R$, now consider $M = M_{n \times n}(R)$ as an $R$-module and $F: A \rightarrow FP(M)$, defined by $F(M) = \{M \cdot N \mid N = N \cdot M\}$ for all $M \in M$. Then $(F, A)$ is a fuzzy soft module over $M$, and $(F, A)$ is an $(\mathcal{R}, A)$-module, which means $(F, \mathcal{R}, A)$ is an FSFS module.

**Definition (3.4):** Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R$-FSFS modules over $M$. Then $(F, \mathcal{R}, A)$ is called an FSFS submodule of $(H, \mathcal{R}, B)$ if

\[(i)\] $A \subseteq B$

\[(ii)\] For all $a \in A$, $F_a$ is a fuzzy submodule of $H_a$

**Theorem (3.5):** Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R$-FSFS modules over $M$. If $F_a \subseteq H_a$ for all $a \in A$, then $(F, \mathcal{R}, A)$ is an FSFS submodule of $(H, \mathcal{R}, B)$.

**Proof.** The proof of the theorem is straightforward.

**Theorem (3.6):** Let $(F, A)$ and $(H, B)$ be two FSFS modules over $M$. Then their intersection $(F, A) \cap (H, B)$ is an FSFS module over $M$.

**Proof.** Let $(F, A) \cap (H, B) = (G, C)$, where $C = A \cap B$. Since the fuzzy soft set $G_c = F_a \cap H_c$ is a fuzzy submodule, for all $c \in C$, $(G, C)$ is a FSFS module over $M$.

**Theorem (3.7):** Let $(F, A)$ and $(H, B)$ be two FSFS modules over $M$. Then $(F, A) \wedge (H, B)$ is an FSFS module over $M$.

**Proof.** By Definition 2.7, we can write $(F, A) \wedge (H, B) = (G, A \times B)$. Since $F_a$ and $H_b$ are fuzzy submodules of $M$, $F_a \wedge H_b$ is a fuzzy submodule of $M$. Thus, $G(a, b) = F_a \wedge H_b$ is a fuzzy submodule of $M$, for all $(a, b) \in A \times B$. Hence, we find that $(F, A) \wedge (H, B)$ is an FSFS module over $M$.

**Theorem (3.8):** Let $(F, A)$ and $(H, B)$ be two FSFS modules over $M$. If $A \cap B = \emptyset$, then $(F, A) \cup (H, B)$ is an FSFS module over $M$.

**Proof.** By Definition 2.5, we can write $(F, A) \cup (H, B) = (G, C)$. Since $A \cap B = \emptyset$, it follows that either $c \in A - B$ or $c \in B - A$ for all $c \in C$. If $c \in A - B$, then $G_b = F_b$ is a fuzzy submodule of $M$, and if $c \in B - A$, then $G_b = H_b$ is a fuzzy submodule of $M$. Hence, $(F, A) \cup (H, B)$ is an FSFS module over $M$.

The following theorem is a generalization of theorems 3.6, 3.7, 3.8.

**Theorem (3.9):** Let $(F, A)$ be an FSFS module over $M$, and let $\{(F_i, A_i)\}_{i \in I}$ be a nonempty family of FSFS submodules of $(F, A)$. Then

\[(i)\] $\bigcap_{i \in I} (F_i, A_i)$ is an FSFS submodule of $(F, A)$.

\[(ii)\] $\bigcup_{i \in I} (F_i, A_i)$ is an FSFS submodule of $(F, A)$.

\[(iii)\] If $A_i \cap A_j = \emptyset$ for all $i, j \in I$, then $\bigcup_{i \in I} (F_i, A_i)$ is an FSFS submodule of $(F, A)$.

**Definition (3.10):** Let $(F, \mathcal{R}_M, A)$ and $(H, \mathcal{R}_N, B)$ be two $R$-FSFS modules over $M$ and $N$, respectively, and let $f: M \rightarrow N$ be a homomorphism of modules, and let $g: A \rightarrow B$ be a mapping of sets. Then we say that $(f, g): (F, \mathcal{R}_M, A) \rightarrow (H, \mathcal{R}_N, B)$ is an FSFS homomorphism of $FSFS$ modules, if the following condition is satisfied:

$f(F_a) = H(g(a)) = H_a(a)$.

We say that $(F, A)$ is an FSFS homomorphic to $(H, B)$.

Note that for all $a \in A, f(M, F_a) \rightarrow (N, H_g(a))$ is a fuzzy homomorphism of fuzzy modules.

To introduce the kernel and image of $FSFS$ homomorphism of $FSFS$ modules, let $\bar{M} = \ker f$. Define $\bar{F}: A \rightarrow PF(\bar{M})$ by $\bar{F}_a = F_a \mid \bar{M}$. Then $(\bar{F}, A)$ is an FSFS module.
over \( \tilde{M} \). It is clear that this module is an FSFS submodule of \((F,A)\).

**Definition (3.11):** \((\tilde{F},A)\) is said to be kernel of \((f,g)\) and denoted by \(\ker (f,g)\).

And, let \(\tilde{B} = g(A)\). Then, for all \(b \in \tilde{B}\), there exists \(a \in A\) such that \(g(a) = b\). Let \(\tilde{N} = \text{Im } f < N\). We define the mapping \(\tilde{H} : \tilde{B} \rightarrow PF(N)\) as \(\tilde{H}(\tilde{b}) = \tilde{B}(g(a))\). Since \((f,g)\) is an FSFS homomorphism, \(f(f_a) = H_a(g(a))\) is satisfied for all \(a \in A\). Then the pair \((\tilde{H},\tilde{B})\) is an FSFS module over \(\tilde{N}\) and \((\tilde{H},\tilde{B})\) is an FSFS submodule of \((H,B)\).

**Definition (3.12):** \((\tilde{H},\tilde{B})\) is said to be image of \((f,g)\) and denoted by \(\text{Im } (f,g)\).

**Proposition (3.13):** Let \((F,A)\) be an FSFS module over \(M\) and \(N\) be an \(R\)-module and \(f : M \rightarrow N\) be a homomorphism of \(R\)-modules. Then \((f(F),A)\) is an FSFS module over \(N\).

**Proof.** If the mapping \(f(F) : A \rightarrow PF(N)\) is defined by \(f(F)_a (y) = \sup \{F_a(x) : f(x) = y\}\), the proof is completed.

**Proposition (3.14):** If \(M\) is an \(R\)-module, \((H,A)\) is an FSFS module over \(N\) and \(f : M \rightarrow N\) is a homomorphism of \(R\)-modules, then \((f^{-1}(H),A)\) is an FSFS module over \(M\).

**Proof.** If the mapping \(f^{-1}(H) : A \rightarrow PF(M)\) is defined by \((f^{-1}(H))_a (x) = H_a(f(x))\), the proof is completed.

It is clear that \((f,1_A) : (f^{-1}(H),A) \rightarrow (H,A)\) is an FSFS homomorphism of FSFS modules.

We now introduce the following lemma:

**Lemma (3.15):** Let \(M\) and \(N\) be an \(R\)-modules and \(f : M \rightarrow N\) be a homomorphism and \((F,A)\) and \((H,A)\) are two FSFS modules over \(M\) and \(N\) respectively.

\(i\) \((f,1_A) : (F,A) \rightarrow (H,A)\), is an FSFS homomorphism if and only if \(\forall a \in A, H_a \geq f(f_a)\).

\(ii\) \((f,1_A) : (F,A) \rightarrow (H,A)\), is an FSFS homomorphism if and only if \(\forall a \in A, f_a \leq f(f_a)\).

**Proof.** \((i)\): \((f,1_A)\) is an FSFS homomorphism which means \(f(f_a) = H(a) \leq H_a\).

Conversely \(f(f_a) = H_a\), then \((f,1_A)\) is an FSFS homomorphism.

\((ii)\) similarly as in \((i)\).

Now we define other algebraic operations over FSFS modules.

**Definition (3.16):**

\[ \prod_{i \in I} (F_i,A_i) \]

is said to be direct product of FSFS modules.

**Proposition (3.17):** Let \(\{(F_i,A_i)\}_{i \in I}\) be a family of FSFS modules over \(\{M_i\}_{i \in I}\) and \(\{(H_i,B_i)\}_{i \in I}\) be a family of FSFS modules over \(\{N_i\}_{i \in I}\) and \((f_i,g_i) : (F_i,A_i) \rightarrow (H_i,B_i)\) be an FSFS homomorphism of FSFS modules for each \(i \in I\). Then

\[ \prod_{i \in I} (F_i,A_i) \rightarrow \prod_{i \in I} (H_i,B_i) \]

is an FSFS homomorphism of FSFS modules.

**Proof.** Since

\[ \left( \prod_i (F_i) \right) \left( \prod_i (H_i) \right) = \prod_i (F_i \ast F_i) = \prod_i (K_{i \in I} g_i) = \left( \prod_{i \in I} K_i \right) \left( \prod_{i \in I} g_i \right) \]

the proof is completed.

**Proposition (3.18):** \(\Pi : \Pi FSFSM \rightarrow FSFSM\) is a functor.

**Proof.** Since by proposition (2.27), \(\Pi : FSFSM \rightarrow FSM\) is a functor, and every FSFS module is an FS module, the proof is completed.

**Definition (3.19):** \((F,A)\) is said to be direct sum of \(\{(F_i,A_i)\}_{i \in I}\) and denoted by \(\oplus_{i \in I} (F_i,A_i)\).

The mapping \(\varphi_j : A_j \rightarrow \prod_{i \in I} A_i\)

is defined as \(\varphi_j(a_j) = \{a_i\} = \begin{cases} a_{i_j} & \text{if } i \neq j \\ a_i & \text{if } i = j \end{cases}\).

Also for embedding mapping \(q_j : M_j \rightarrow \prod_{i \in I} M_i\), \((q_j,\varphi_j) : (F_j,A_j) \rightarrow (F,A)\) is an FSFS homomorphism of FSFS modules.
Proposition (3.20): Let \( \{(F_i, A_i)\}_{i \in I} \) and \( \{(H_j, B_j)\}_{j \in J} \) be family of FSFS modules over \( \{M_i\}_{i \in I} \) and \( \{N_j\}_{j \in J} \) respectively, and let \( (f_i, g_j) : (F_i, A_i) \rightarrow (H_j, B_j) \) be an FSFS homomorphism of FSFS modules. Then
\[
\left( \prod_{i \in I} f_i \right) : \prod_{i \in I} (F_i, A_i) \rightarrow \prod_{i \in I} (H_i, B_i)
\]
is an FSFS homomorphism of FSFS modules.

Proposition (3.21): \( \oplus : \text{IFSFM} \rightarrow \text{FSFM} \) is a functor.

Proof: since by proposition (2.30), \( \oplus : \text{IFSFM} \rightarrow \text{FSM} \) is a functor, and every FSFS module is an FS module, the proof is completed.

Theorem (3.22): If \( \{(F_i, A_i)\}_{i \in I} \) is a family of FSFS modules over \( \{M_i\}_{i \in I} \), then
\[
\prod_{i \in I} (F_i, A_i)
\]
is an FSFS module over \( \prod_{i \in I} M_i \).

Proof: Define
\[
F: \prod_{i \in I} A_i \rightarrow \prod_{i \in I} M_i
\]
by \( F(\{a_i\}) = \gamma \prod_{i \in I} p_i^{-1}(F_i) a_i \) where
\[
p_i: \prod_{i \in I} M_i \rightarrow M_i
\]
is a projection mapping. Since
\[
p_i^{-1}(F_i) a_i: \prod_{i \in I} M_i \rightarrow [0,1]
\]
is an FSFS module over \( \prod_{i \in I} M_i \),

for all \( i \in I \),
\[
\bigwedge_{i \in I} p_i^{-1}(F_i) a_i
\]
is also an FSFS module over \( \prod_{i \in I} M_i \).

Theorem (3.23): If \( \{(F_i, A_i)\}_{i \in I} \) is a family of FSFS modules over \( \{M_i\}_{i \in I} \), then
\[
\oplus_{i \in I} (F_i, A_i)
\]
is an FSFS module over \( \oplus_{i \in I} M_i \).

Proof. Define
\[
F: \prod_{i \in I} A_i \rightarrow \oplus_{i \in I} M_i
\]
for all \( \{a_i\} \in \prod_{i \in I} A_i \)
by
\[
F(\{a_i\}) = \bigwedge_{i \in I} j_i(F_i) a_i
\]
where \( j_i : M_i \rightarrow \oplus_{i \in I} M_i \) is an embedding mapping.
Since \( f_i : M_i \rightarrow \oplus_{i \in I} M_i \) is an FSFS submodule over \( \oplus_{i \in I} M_i \) for all \( i \in I \), \( F(\{a_i\}) \) is also an FSFS module over \( \oplus_{i \in I} M_i \).

Lemma (3.24): 1) Given modules \( \{M_i\}_{i \in I} \) and \( N \) and a family of \( R \)–homomorphisms \( A = \{ f_i : M_i \rightarrow N \}_{i \in I} \). If \( \{(F_i, A_i)\}_{i \in I} \) are FSFS modules over \( \{M_i\}_{i \in I} \), then there exist an FSFS module
\[
(H, \prod_{i \in I} A_i)
\]
over \( N \) such that for all \( i \in I \),
\[
f_i : (F_i, A_i) \rightarrow (H, \prod_{i \in I} A_i)
\]
is an FSFS homomorphism of FSFS modules.

2) Given modules \( M \) and \( \{N_i\}_{i \in I} \) and a family of \( R \)–homomorphisms \( B = \{ g_i : M \rightarrow N_i \}_{i \in I} \). If \( \{(H_i, B_i)\}_{i \in I} \) are FSFS modules over \( \{N_i\}_{i \in I} \), then there exist an FSFS module \( (F, \prod_{i \in I} A_i) \) over \( M \) such that for all \( i \in I \),
\[
g_i : (F_i, A_i) \rightarrow (H_i, B_i)
\]
is an FSFS homomorphism of FSFS modules.

Proof. 1) Define : \( H: \prod_{i \in I} A_i \rightarrow N \) by \( F(\{a_i\}) = \bigwedge_{i \in I} f_i(F_i) a_i \).

2) Define : \( F: \prod_{i \in I} A_i \rightarrow M \) by \( F(\{a_i\}) = \bigwedge_{i \in I} g_i^{-1}(F_i) a_i \) for all \( \{a_i\} \in \prod_{i \in I} A_i \).

We define the concepts of quotient module, product and coproduct operations in the category of FSFS modules.

Corollary (3.25): If \( (F, A) \) is a FSFS module over \( M \) and \( N \) is a submodule of \( M \), \( i : N \rightarrow M \)
is an embedding mapping, then \((t^{-1}(F), A)\) is an FSFS module over \(N\).

**Corollary (3.26):** If \((F, A)\) is an FSFS module over \(M\) and \(p: M \rightarrow M/N\) is a canonical projection, then \((p(F), A)\) is an FSFS module over quotient module \(M/N\).

If \(\{t_i \mid i \in I\}\) is a family of FSFS modules over the family of modules \(\{M_i\mid i \in I\}\), then we can define the product and coproduct of these families by \(\prod_{i \in I} (F_i, A_i)\) and \(\bigoplus_{i \in I} (F_i, A_i)\) respectively.

**Theorem (3.27):** The category of FSFS modules has zero objects, sums, product, kernel and cokernel.

**Theorem (3.28):** Let \((F, A)\) and \((G, B)\) be two FSFS modules over \(M\) and \(N\), respectively, and \(F \otimes G: A \times B \rightarrow M \otimes N\). Then \((F \otimes G, A \times B)\) is an FSFS module over \(M \otimes N\).

**Proof.** Let \((F \otimes G)(a, b) = F_a \otimes G_b, \forall (a, b) \in A \times B\) \([7]\). \(\forall (a, b) \in A \times B, (M, F_a)\) and \((N, G_b)\) are FSFS modules. From \([12]\), \(F_a \otimes G_a\) is a fuzzy submodule over \(M \otimes N\). Then \((F \otimes G, A \times B)\) is an FSFS module over \(M \otimes N\).

**Definition (3.29):** \((F \otimes G, A \times B)\) is said to be tensor product of \((F, A)\) and \((G, B)\), and denoted by \((F, A) \otimes (G, B)\).

**References:**