Modules With Chain Conditions On $\delta$-Small Submodules

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Abstract:
Let $R$ be an associative ring with identity and $M$ be unital non zero $R$-module. A submodule $N$ of a module $M$ is called a $\delta$-small submodule of $M$ (briefly $N<<\delta M$) if $N+X=M$ for any proper submodule $X$ of $M$ with $M/X$ singular, we have $X=M$.

In this work, we study the modules which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on this kind of submodules. Then we generalize this conditions into the rings, in the last section we get same results on $\delta$-supplement submodules and we discuss some of these results on this types of submodules.

Keywords: $\delta$-small submodule, $\delta$-supplement submodules, c-singular submodule.

المقاسات التي تحقق خاصية السلسلة للمقاسات الجزئية $\delta$ الصغيرة

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الخلاصة:
لتكن $R$ حلقة تجميعية ذات عنصر محايد وليكن $M$ مقاسا احاديا غير صفري يمني معرفا على $R$. مقاسا $N$ من $M$ يقال بأنه $\delta$ صغير إذا كان $N+X=M$ لكل مقاس جزئي $X$ من $M$ بحيث $M/X$ منفردا فان $X=M$ في هذا البحث سنقوم بدراسة هذا النوع من المقاسات الجزئية والمقاسات التي تحقق خاصيتي السلسلة على المقاسات الجزئية $\delta$ صغيرة، كذلك قمنا بتعريف هذه الشروط على الحقول، وفي الجزء الأخير حصلنا على بعض النتائج عن المقاسات الجزئية $\delta$-المكملة ووضوح بعض نتائجها.

1. Introduction
Let $R$ be an associative ring with identity and $M$ is a non zero unital right $R$-module. A submodule of $R$-module $A$ is called essential ($A\subseteq_e M$) if every non zero submodule of $M$ has non intersection with $A$. $M$ is called singular module if $M$ is a submodule $X$ of $M$ is called $c$-singular ($X\subseteq_c M$) if $X/M$ singular module.

An ideal $I$ of a ring $R$ is $\delta$-small ideal if $I$ is $\delta$-small $R$-submodule of $R$.

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Remark 1.1-[3]
1-let A be submodule of R-module M if A ≤ M then A ≤_c.s M.
2- Let M and N be R-modules and
\[ f : M \to N \text{ be an epimorphism if } A \subseteq_c.s M \text{ then } f(A) \subseteq_c.s N. \]

3- Let A and B be submodules of R-module M if A \subseteq_c.s M and B \subseteq_c.s M, then (A \cap B) \subseteq_c.s M.
4- Every submodule of a singular module is c-singular.

Lemma 1.2 [2]: Let M be a module,
1) For submodules N, K, L of M with K ⊆ N then
da) N <<_c.s M if and only if K <<_c.s M and N/K <<_c.s M/K
b) N + L <<_c.s M if and only if N <<_c.s M and L <<_c.s M.
2) If K <<_c.s M and f : M \to N a homo then f(K) <<_c.s N.
3) If K1 ⊆ M1 \subseteq M, K2 ⊆ M2 \subseteq M, and M = M1 \oplus M2 then K1 \oplus K2 <<_c.s M1 \oplus M2 if and only if K1 <<_c.s M and K2 <<_c.s M2.
4) Let A <<_c.s B, if A <<_c.s M and B is a direct summand then A <<_c.s B.

In [4], If N and L be submodules of a module M, N is called a δ-supplement of L in M if M = N + L and N \cap L <<_c.s M, and if every submodules of M has a δ-supplement in M, Then M is called a δ-supplement module.

An R-module M is said to satisfy the ascending chain condition (a.c.c.) on small submodules. respectively descending chain condition (d.c.c.) on small submodules if every ascending descending chain of small submodules K₁ ⊆ K₂ ⊆ K₃ ⊆ ... ⊆ Kₙ ... respectively K₁ ⊇ K₂ ⊇ K₃ ⊇ ... ⊇ Kₙ ... Terminates[5].

In this work, we study the modules which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on δ-small submodules. Then we generalize these conditions into the rings. In the last section we get some results on δ-supplement submodules and we discuss some of these results on this types of submodules.

2. Modules with chain conditions on a δ-small submodules

In this section, we introduce the definition of module which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on δ-small submodules as a generalization of chain condition (a. c. c.) and descending chain condition (d. c. c.) on small submodules [5] and we study the relation between the ring that satisfies (a. c. c.) and descending chain condition (d. c. c.) on δ-small ideals.

Definition (2.1): An R-module M is said to satisfy the ascending chain condition (a.c.c.) on δ-small submodules, respectively descending chain condition (d.c.c.) on δ-small submodules if every ascending (descending) chain of δ-small submodules K₁ ⊆ K₂ ⊆ K₃ ⊆ ... ⊆ Kₙ ... respectively K₁ ⊇ K₂ ⊇ K₃ ⊇ ... ⊇ Kₙ ... terminates.

Since every small submodule is δ-small submodule, The following is clear

Remark (2.2): If M satisfy the a.c.c.(d.c.c.) on δ-small submodules then M satisfy the a.c.c.(d.c.c.) on small submodules.

Proposition (2.3): Let M₁ and M₂ be two R-modules and R = annM₁ + annM₂. Then M₁ ⊗ M₂ satisfies a.c.c.(d.c.c.) on δ-small submodules if M₁ and M₂ satisfies a.c.c.(d.c.c.) on δ-small submodules.
Proof: Since $R=\text{ann}M_{1}+\text{ann}M_{2}$, let $N_{1} \oplus K_{1} \subseteq N_{2} \oplus K_{2} \subseteq N_{3} \oplus K_{3} \subseteq \ldots \subseteq N_{n} \oplus K_{n}$ be ascending chain on $\delta$-small submodules of $M_{1} \oplus M_{2}$, hence, $N_{1} \subseteq N_{2} \subseteq N_{3} \subseteq \ldots \subseteq N_{n}$ is ascending chain on $\delta$-small submodules of $M_{1}$ and $K_{1} \subseteq K_{2} \subseteq K_{3} \subseteq \ldots \subseteq K_{n}$ be ascending chain on $\delta$-small submodules of $M_{2}$. Since $M_{1}$ and $M_{2}$ satisfies a.c.c. on $\delta$-small submodules then $\exists \ t, r \in Z^{+}$ such that $N_{t}=N_{r+1}=\ldots \forall \ i=1,2,3 \ldots$ and $K_{r}=K_{r+1}=\ldots \forall \ i=1,2,3 \ldots$ take $s=\max \{t,r\}$, hence $N_{s}+K_{s}=N_{s+1}+K_{s+1}=\ldots \forall \ i=1,2,3$.

Conversely, let $N_{1} \subseteq N_{2} \subseteq N_{3} \subseteq \ldots \subseteq N_{n}$ be ascending chain on $\delta$-small submodules of $M_{1}$ and since $N_{r} \subseteq N_{r+1}$, it is a contradiction since $L_{r} \neq 0$.

Let $N_{s}+K_{s}=N_{s+1}+K_{s+1}=\ldots \forall \ i=1,2,3 \ldots$ take $s=\max \{t,r\}$, hence $N_{s}+K_{s}=N_{s+1}+K_{s+1}=\ldots \forall \ i=1,2,3$.

Recall that an $R$-module $M$ is called multiplication if $M=MI$ for some ideal $I$ of $R$. The following proposition gives a relation between $\delta$-small ideals and $\delta$-small submodules of a finitely generated faithful multiplication modules.

Proposition (2.4): Let $M$ be a finitely generated faithful multiplication $R$-module, and let $N=MI$ for some ideal $I$ of $R$ then $N$ is $\delta$-submodule of $M$ iff $I$ is $\delta$-small ideal in $R$.

Proof: Assume $N$ is $\delta$-small in $M$, and $N=MI$, let $I+J=R$, for some c-singular ideal $J$ of $R$, then $M+J=M=MI$ and $N=M+J$. Since $M=MI$ a faithful multiplication module, hence $N=MI$. Similarly, let $N=MI$, and $I=J$.

Conversely, let $N=MI$ for some $I$ of $R$, then $N=MI$. Let $M=MI$ then $N=MI$.

Corollary (2.5). Let $M$ be a finitely generated faithful multiplication $R$-module, and let $N=MI$ for some ideal $I$ of $R$ then $N \subseteq M$. Let $N=MI$ for some ideal $I$ of $R$ then $N \subseteq M$.

Corollary (2.6) Let $M$ be a finitely generated faithful multiplication $R$-module, then $R$ satisfies a.c.c. on c-singular ideal if and only if $M$ satisfies a.c.c. on c-singular submodules.

Proof: $I_{1} \subseteq I_{2} \subseteq \ldots \subseteq I_{k}$ be an ascending chain of c-singular ideals in $R$ then by Corollary 2.5 $M_{1} \subseteq M_{2} \subseteq \ldots \subseteq M_{k}$ is an ascending chain of c-singular submodules of $M$. Since $M$ satisfies a.c.c. on c-singular submodules then $\exists \ K \in N_{k}$, such that $M_{K}=M_{K+1}$, But $M$ is a finitely generated faithful module, then $I_{k}=I_{k+1} \ldots \forall k=1,2,3$.

Conversely, let $N_{1} \subseteq N_{2} \subseteq \ldots \subseteq N_{k}$ be an ascending chain of c-singular submodules of $M$. Since $M$ is a multiplication $R$-module, then $N_{i}=I_{i}M$, for some ideal $I_{i}$ of $R$ for all $i$. Hence $M_{I_{1}} \subseteq M_{I_{2}} \subseteq \ldots \subseteq M_{I_{k}}$. But $M$ is finitely generated then by Corollary 2.5.
I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots \text{ is an ascending chain of c-singler ideals in } R. \text{ Since } R \text{ satisfies a.c.c on c-singler ideal }, \exists K \in N, \text{ such that } I_k = I_{k+1} = \ldots, \text{ hence } M I_k = M I_{k+1} = \ldots \text{ which implies } N_k = N_{k+1} = \ldots, \text{ that is } M \text{ satisfies a.c.c on c-singler submodule of } M.

The following results are sequences of this proposition.

**Corollary (2.7):** Let } M \text{ be a finitely generated faithful multiplication } R \text{-module, then } R \text{ satisfies a.c.c. on } \delta \text{-small ideal if and only if } M \text{ satisfies a.c.c. on } \delta \text{-small submodules.}

Proof: Let } N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots \text{ be an ascending chain of small submodule of } M. \text{ Since } M \text{ is a multiplication } R \text{-module, then } N_i = I_i M, \text{ for some ideal } I_i \text{ of } R \text{ for all } i.

Hence } M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots \text{ But } M \text{ is finitely generated then by proposition (2.4)} I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots \text{ be an ascending chain of } \delta \text{-small ideals in } R. \text{ Since } R \text{ satisfies a.c.c on } \delta \text{-small ideal, then } \exists K \in N, \text{ such that } I_k = I_{k+1} = \ldots, \text{ hence } M I_k = M I_{k+1} = \ldots \text{ which implies } N_k = N_{k+1} = \ldots, \text{ that is } M \text{ satisfies a.c.c on } \delta \text{-small submodules. Conversely, let } I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots \text{ be an ascending chain of } \delta \text{-small ideals in } R, \text{ then by Proposition (2.4)} M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots \text{ is an ascending chain of } \delta \text{-small submodule of } M. \text{ Since } M \text{ satisfies a.c.c. on } \delta \text{-small submodules then there exists } K \in N, \text{ such that } M I_k = M I_{k+1} = \ldots \text{ But } M \text{ is a finitely generated faithful module then } I_k = I_{k+1} = \ldots [7]. \text{ Thus } R \text{ satisfies a.c.c on } \delta \text{-small ideals of } R.

**Proposition (2.8):** Let } M \text{ be an } R \text{-module, satisfies a.c.c. on } \delta \text{-small submodules. Then } M \text{ is } \delta \text{-small submodule of } M \text{ and } M A \text{ satisfies a.c.c. on } \delta \text{-small submodules of } M.

Proof: Let } A_i \subseteq A_j \subseteq \ldots \text{ be a.c.c. on } \delta \text{-small submodules of } M A \text{ then } A_i \subseteq A_j \subseteq \ldots \text{ But } A \text{ is } \delta \text{-small submodule and } A_i \ll_\delta M \text{ then } A_i \ll_\delta M \forall I \text{ (Lemma 1.2)} \text{ thus } A_i \subseteq A_j \subseteq \ldots \text{ is an ascending chain of } \delta \text{-small submodule of } M. \exists K \in N, \text{ such that } A_n = A_{n+1} = \ldots \text{ thus } M A \text{ satisfies a.c.c on } \delta \text{-small submodules Similar proof for (d.c.c.). Hence we have the following result:}

**Theorem (2.9):** Let } M \text{ be a finitely generated faithful multiplication } R \text{-module, then the following are equivalent.}

1) } M \text{ satisfies a.c.c (d.c.c) on } \delta \text{-small submodules.}
2) } R \text{ satisfies a.c.c (d.c.c) on } \delta \text{-small ideals.}
3) } S = \text{End}_R(M) \text{ satisfies a.c.c (d.c.c) on } \delta \text{-small ideals.}
4) } M \text{ satisfies a.c.c (d.c.c) on } \delta \text{-small submodules as } S \text{-module.}

Proof: (1) \Rightarrow (2) \text{ By Cor (2.7)}
(2) \Rightarrow (3) \text{ since } M \text{ is a finitely generated faithful multiplication } R \text{-module, then } R \cong S \text{ hence } R \text{ satisfies a.c.c (d.c.c) } S = \text{End}_R(M) \text{ satisfies a.c.c (d.c.c) on } \delta \text{-small ideals.}
(3) \Rightarrow (4) \text{ By Cor (2.7)}
(4) ⇒ (1) By Cor (2.7) R satisfies a.c.c (d.c.c) on δ-small ideals. R=σ[7] hence R satisfies a.c.c(d.c.c) on δ-small ideals and by cor (2.7) M satisfies a.c.c (d.c.c) on δ-small submodules.

3. Modules with chain conditions on δ-supplement submodule

It is known that Rad(M) is the sum of all small submodules of M. In [2] Zhou introduced the δ(M) as a generalization of Rad(M).

**Definition 3.1 [2]:** Let ρ be the class of all singular simple modules. For a module M, let δ(M) = {I | N ⊆ M, M/N ∈ ρ} be the reject M of ρ.

**Lemma 3.2:** [2, Lemma 1.5] Let M and N be R-modules

1) δ(M) = Σ{ L ⊆ M / L is δ-small submodule of M }
2) If f : M → N is an R-homomorphism then f(δ(N)) ⊆ δ(N). Therefore δ(M) is a fully invariant submodule of M and M. δ(R) ⊆ δ(M)
3) If M = Σi=1Mi, then δ(M) = Σi=1 δ(Mi)
4) If every proper submodule of M is contained in maximal submodule of M then δ(M) is unique largest δ-small submodule of M.
5) Let m ∈ M then Mm << δ M iff m ∈ δ(M).
6) An arbitrary sum of δ-small submodules of M is δ-small submodule of M iff δ(M) << δ M.

**Remark (3.3):** Let M be a finitely generated R-module. Then for any submodule A of M, A is δ-small iff A ⊆ δ(M).

**Proof:** Clear from Lemma 3.2 and [1, Th.2.3.11].

**Proposition (3.4):** Let M be an R-module then the following are equivalent

a) M satisfies a.c.c (d.c.c) on δ-small submodules
b) Every non-empty collection of δ-small submodules possesses a maximal (minimal) member.

**Proof:** Clear.

**Proposition (3.5):** Let M be an R-module then M satisfies a.c.c on δ-small submodules if and only if δ(M) is δ-small and every δ-small submodule is finitely generated.

**Proof:** Assume M satisfies a.c.c on δ-small submodules. Let μ = {B : B is a finite sum of δ-small submodules of M} then μ is non-empty collection of δ-small submodules by Lemma 1.2 so by Prop.2.4 μ has maximal element say K hence K is δ-small submodule of M then K ⊆ δ(M). [Lemma 3.2,6]. Suppose that there exists x ∈ δ(M) and x ∉ K hence Rx is δ-small submodule of M [Lemma 3.2,5] so K+Rx is δ-small submodule thus K+Rx ⊆ μ and K ⊆ K+Rx this contradiction the maximality of K then K=δ(M) thus δ(M) is δ-small submodule. Consider any δ-small submodule A of M and let G = {B : B is finitely generated δ-small submodule of M, B ⊆ M} since the zero submodule is contained in G, G ≠ φ. By Prop.3.4, G has a maximal element say K, we claim that K=A, Since K ∈ G, K is finitely generated and K ⊆ A.

If K ≠ A then ther exist x ∈ A, x ∉ K, hence K+Rx is member of G, contradicting K is condensation maximality of K then K=A and then A is finitely generated.

For the converse, consider I1 ⊆ I2 ⊆ I3 ⊆ … ⊆ Ik ⊆ … an ascending chain of δ-small submodules, let I = ∪i=1∞ Ii then I ⊆ δ(M) since for every i=1,2,3 Ii ⊆ δ(M). But δ(M) is δ-small submodule of M so I is δ-small submodule of M thus I is finitely generated I_=Rx1+Rx2+…+Rxn now each xi ∈ Ii for every i so there exist m such that x1,x2,…,xm ∈ Im . But this implis that I = Im so I_m = I_m+1 = ….. thus M satisfies a.c.c on δ-small submodules.

From remark (3.3) and similar proof of prop.(3.5) we get the following

**Corollary (3.6):** Let R be a ring then R satisfies a.c.c on δ-small ideal if and only if every δ-small ideal is finitely generated.

Let N and L be submodules of a module M. N is called a supplement of L in M if M=N+L and
In [4] If N and L be submodules of a module M. N is called a \( \delta \)-supplement of L in M if M=N+L and N \( \cap \) \( L \leq \) N, and if every submodules of M has a \( \delta \)-supplement in M. Then M is called a \( \delta \)-supplement module. R is called a \( \delta \)-supplement if it is supplement as R-module. It is clear that every supplement submodule is a \( \delta \)-supplement but the converse is not true [4].

**Proposition (3.7):** Let N and L be submodules of a finitely generated faithful multiplication R-module M such that N= M I, and L=NJ for some ideals I, J of R then N is \( \delta \)-supplement submodule of L in M iff I is \( \delta \)-supplement ideal of J in R.

**Proof:** If N is \( \delta \)-supplement submodule of L, then M= N+L and N \( \cap \) \( L \leq \delta \) M, then M I + M J = M, MI \( \cap \) MJ \( \leq \delta \) M hence M(I+J) = M and M(I \( \cap \) J) \( \leq \delta \) M then R=I+J, by prop. 2.4 I \( \cap \) J \( \leq \delta \) I hence I is \( \delta \)-supplement ideal of J in R. As the same proof the converse is true.

**Corollary (3.8):** Let M be a finitely generated faithful multiplication R-module, then R satisfies a.c.c. (d.c.c.) on \( \delta \)-supplement ideal if and only if M satisfies a.c.c. (d.c.c.) on \( \delta \)-supplement submodules.

**Proof:** Let \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots \) be an ascending chain of \( \delta \)-supplement ideals of J_i in R, then M \( I_1 \subseteq M \subseteq I_2 \subseteq M \subseteq I_3 \subseteq \ldots \subseteq M \subseteq \ldots \) is an ascending chain of \( \delta \)-supplement submodule of J_i in M in M unless i=1,2,.. by prop 3.7 then \( \exists K \in N \), such that M \( I_k = M I_{k+1} = \ldots \). But M is a finitely generated faithful module, then \( I_k = I_{k+1} = \ldots \) [7]. Thus R satisfies a.c.c. on \( \delta \)-supplement ideals of R.

Conversely, let \( N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots \) be an ascending chain of \( \delta \)-supplement submodules of \( L_i \) \( \cap \) \( i=1,2,\ldots \). Since M is a multiplication R-module, then \( N_i = M I_i \) and \( L_i = M J_i \) where \( J_i \), \( I_i \) ideals of \( R \) for all \( i \) by prop. 4.1 \( I_i \) are \( \delta \)-supplement ideals of \( J_i \) in R, hence \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots \) is an ascending chain of \( \delta \)-supplement ideals of \( J_i \) in R. Since R satisfies a.c.c on \( \delta \)-supplement ideal, then \( \exists K \in N \), such that \( I_k = I_{k+1} = \ldots \), hence M \( I_k = M I_{k+1} = \ldots \) which implies \( N_k = N_{k+1} = \ldots \), that is M satisfies a.c.c. on \( \delta \)-supplement submodules.

The same argument for d.c.c. condition hence omitted.

**References**