Variational Iteration Method for Solving Multi-Fractional Integro Differential Equations

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Abstract:
In this paper, we present an approximate method for solving integro-differential equations of multi-fractional order by using the variational iteration method.

First, we derive the variational iteration formula related to the considered problem, then prove its convergence to the exact solution. Also we give some illustrative examples of linear and nonlinear equations.

Keywords: Fractional calculus, integro-differential equations, variational iteration method.

الطريقة التغايرية التكرارية لحل المعادلات التفاضلية–التكاملية ذات الرتب الكسورية المتعددة

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الخلاصة:
في هذا البحث، تم تقديم طريقة تقريبية لحل المعادلات التفاضلية التكاملية ذات الرتب الكسورية المتعددة بأستخدام الطريقة التغايرية التكرارية.

لقد تم اشتقاق الصيغة التكرارية لحل المسألة المقترحة ومن ثم برهان تقارب متتابعة الحلول الى الحل المضبوط مع إعطاء بعض الامثلة التوضيحية الخطية وغير الخطية لحل هكذا نوع من المعادلات.

1. Introduction:

The fractional integro-differential equations is a special kind of equations collecting integral equations and fractional calculus and in recent years, there has been a growing interest in the integro-differential equations, since many mathematical formulations of physical phenomena, such as nonlinear functional analysis and their applications in the Theory of Engineering, Mechanics, Physics, Chemical Kinetics, Astronomy, Biology, Economics, Potential Theory and Electrostatistics contain integro-differential equations, [1-3].

The variational iteration method (VIM) was proposed originally by Ji-Huan He [4]. An elementary introduction to the variational iteration method and some new developments, as well as, to new interpretations, can be found in [5,6]. This method has been advantageously employed for solving various kinds of nonlinear problems. It has been successfully applied to parabolic partial differential equations [7], to nonlinear systems of second-order boundary value problems [8], to multi-pantograph delay equations [9], to heat-like and wave-like equations with variable coefficients [10], to neutral functional-differential equation with proportional delaysand to other problems [11], recently, Wadeá

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in 2012 used variational iteration method for solving fractional order integro-differential equations [12].

In this paper, we present an approximate method for solving integro-differential equations of fractional order of the form:

$$ {}^{C}D^{\alpha} y(x) = f(x) + I^{\beta} k_1[y(x)] + I^{\gamma} k_2[y(x)] \tag{1} $$

where k_1 , k_2 are given continuous functions, $0 \le \alpha \le 1$, β , $\gamma > 0$, $x \in [a, b]$, $y(a) = A \in R$ and ${}^{C}D^{\alpha}$ refers to the Caputo fractional derivative of order α, while I^{β} and I^{γ} refers to the fractional integrals of order β and γ respectively.

The fractional integro-differential equation could be considered as an important type of integro-differential equations, where the differentiation and the integration appears in the equation is of non-integer order.

2. Basic Concepts:

In this section, some basic fundamental concepts and definitions concerning with fractional calculus and calculus of variation will be introduced for completeness purpose.

2.1 Fractional derivative:

There are various types of definitions for the fractional order derivatives of order q > 0, the most commonly used definitions among various definitions of fractional order derivatives of order q>0 are the Riemann-Liouville and Caputo formula, n this paper we used Caputo fractional derivative, which is defined to be [13]:

$$ {}_{a}^{C}D_{x}^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-s)^{m-\alpha-1}u^{(m)}(s)\,ds \tag{2} $$

where $m-1 < \alpha \le m$, $m \in N$, $x > 0$ and Γ refers to the gamma function.

2.2 Fractional Integral:

As in fractional derivatives, there are many literatures introduces different definitions of fractional integration, in this paper we used the definition of Riemann-Liouville fractional integral, which is defined for the right hand side integral by [14]:

$$ {}_{a}I_{x}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-s)^{\alpha-1}u(s)\,ds\,,\ \alpha > 0,\ a \in R \tag{3.a} $$

and the left hand side fractional integral:

$$ {}_{x}I_{b}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}u(s)\,ds\,,\ \alpha > 0,\ b \in R^{+} \tag{3.b} $$

3. Variational Iteration Method, [4]:

To illustrate the basic idea of the VIM, we consider the following general non-linear equation given in operator form:

$$ L(u(x)) + N(u(x)) = g(x),\ x \in [a, b] \tag{4} $$

where L is a linear operator, N is a nonlinear operator and g(x) is any given function which is called the non-homogeneous term.

Now, rewrite eq.(4) in the form:

$$ L(u(x)) + N(u(x)) - g(x) = 0 \tag{5} $$

and let u_n be the n^{th} approximate solution of eq. (5), then it follows that:

$$ L(u_n(x)) + N(u_n(x)) - g(x) \ne 0 \tag{6} $$

and therefore the correction functional for (4), is given by:

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^{x} \lambda(s)\{L(u_n(s) + N(\tilde{u}_n(s)) - g(s)\}ds \tag{7}$$

where λ is the general Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the n^{th} approximation of the solution u and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$, [1].

To solve eq. (7) by the VIM, we must first evaluate the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximation $u_n(x)$, n = 0, 1, …; of the solution u(x) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x)$. The $zero^{th}$ approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions with λ determined, then several approximations $u_n(x)$, n = 0, 1, …; follows immediately, and consequently the exact solution may be arrived since:

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{8}$$

4. Variational Iteration Method for Solving Multi-Fractional Order Integro-Differential Equations:

Consider the fractional integro-differential equation (1), which may be rewritten as:

$${}^{C}D^{\alpha}y(x) - f(x) - I^{\beta}k_1(y(x)) - I^{\gamma}k_2(y(x)) = 0 \tag{9}$$

Multiply eq.(9) by a general Lagrange multiplier λ, yields to:

$$\lambda(s)\{{}^{C}D^{\alpha}y(x)-f(x)-I^{\beta}k_1(y(x))-I^{\gamma}k_2(y(x))\}=0 \tag{10}$$

Now, take I^{α} to the both sides of eq.(10), which give:

$$I^{\alpha}[\lambda(s)\{{}^{C}D^{\alpha}y(x) - f(x) - I^{\beta}k_1(y(x)) - I^{\gamma}k_2(y(x))\}] = 0 \tag{11}$$

Then, the correction functional for eq.(1) will be read as follows:

$$y_{n+1}(x) = y_n(x) + I^{\alpha}\lambda(s)\{{}^{C}D^{\alpha}y_n(x) - f(x) - I^{\beta}k_1(y_n(x)) - I^{\gamma}k_2(y_n(x)\ \} \tag{12}$$

In this case the value of λ may not be evaluated easily from eq.(12), which will give a functional with fractional integrals. Therefore, the approximation of the correctional functional can be expressed as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\{\frac{d^m y_n(s)}{ds^m} - f(s) - I^{\beta}k_1(\tilde{y}_n(s)) - I^{\gamma}k_2(\tilde{y}_n(s))\}ds \tag{13}$$

Thus, by taking the first variation with respect to the independent variable y_n and noticing that $\delta y_n(0) = 0$, yields to:

$$\delta y_{n+1}(x) = \delta y_n(x) + \int_0^x \lambda(s)\{\frac{d^m y_n(s)}{ds^m} - f(s) - \frac{1}{\Gamma(\beta)}\int_a^s (s-t)^{\beta-1}k_1(\tilde{y}_n(t))dt - \frac{1}{\Gamma(\gamma)}\int_a^s (s-t)^{\gamma-1}k_2(\tilde{y}_n(t))dt\}ds$$

… (14)

where $\tilde{y}_n$ is considered as a restricted variation, which means that $\delta \tilde{y}_n = 0$, and consequently eq.(14) with m = 1 will be reduced to:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(s) y_n'(s)\, ds \tag{15}$$

Hence, using the method of integration by parts on eq.(15) will give the following formula:

$$\delta y_{n+1}(x) = \delta y_n(x) + \lambda(s)\, \delta y_n(s)\Big|_{s=x} - \int_0^x \lambda'(s)\delta y_n(s)\,ds$$

and then:

$$\delta y_{n+1}(x) = (1 + \lambda(s))\delta y_n(x) - \int_0^x \lambda'(s)\delta y_n(s)\,ds = 0$$

As a result, the following stationary conditions are obtained:

$$\lambda'(s) = 0,\ 1 + \lambda(s)\Big|_{s=x} = 0$$

and solving the last ODE will give the general Lagrange multiplier can be defined in the following form:

$$\lambda(s) = -1$$

Hence, substituting $\lambda(s) = -1$ into the correction functional (12), will give the following variational iteration formula:

$$y_{n+1}(x) = y_n(x) - I^{\alpha}\{D^{\alpha}y_n(x) - f(x) - I^{\beta}k(y_n(x)) - I^{\gamma}k_2(y_n(x)\} \quad (16)$$

5. Convergence Anaysis:

In this section, we study the convergence of the variational iteration method, according to the alternative approach of VIM presented in the previous section. The main results are proposed in the following theorem:

<u>Theorem (1)</u>

Let $y\in (C^2[0, T], \|\,.\,\|_{\infty})$ be the exact solution of the integro-differential equation of fractional order (1) and $y_n\in C^2[0, T]$ be the obtained solution of the sequence defined by eq.(16). If

$E_n(x) = y_n(x) - y(x)$ and let

$k(y(x))= \Gamma(\alpha + \gamma)\,(x-s)^{\beta}\, k_1(y(s)) +\Gamma(\alpha + \beta)\,(x-s)^{\gamma}\, k_2(y(s))$

satisfies Lipschitz condition with constant L, such that $L < \Gamma(\alpha + \beta)\Gamma(\alpha + \gamma)$, then the sequence of approximate solutions $\{y_n\}$, n = 0, 1, ...; converges to the exact solution y.

Proof:

Consider the integro-differential equation of fractional order:

${}^{C}D^{\alpha}y(x) = f(x) + I^{\beta}k_1(y(x)) + I^{\gamma}k_2(y(x)), u(0) = u_0$

where the approximate solution using the VIM is given by:

$$y_{n+1}(x) = y_n(x) - I^{\alpha}\{{}^{C}D^{\alpha}y_n(x) - f(x) - I^{\beta}k(y_n(x)) - I^{\gamma}k_2(y(x))\} \quad (17)$$

Since y is the exact solution of the integro-differential equation of fractional order, hence:

$$y(x) = y(x) - I^{\alpha}\{{}^{C}D^{\alpha}y(x) - f(x) - I^{\beta}k_1(y(x)) - I^{\gamma}k_2(y(x))\} \quad (18)$$

Hence, subtracting eq.(18) from eq.(17) yields to:

$$E_{n+1}(x) = E_n(x) - I^{\alpha}\{{}^{C}D^{\alpha}E_n(x) - f(x) + f(x) - I^{\beta}[k_1(y_n(x)) - k_1(y(x))] - I^{\gamma}[k_2(y_n(x)) - k_2(y(x)]\}$$
$$= E_n(x) - I^{\alpha}\{{}^{C}D^{\alpha}E_n(x) - I^{\beta}[k_1(y_n(x)) - k_1(y(x))] - I^{\gamma}[k_2(y_n(x)) - k_2(y(x)]\}$$

Using the properties of fractional calculus since,

$$I^{\alpha}\, D^{\alpha}E_n(x) - E_n(x) - C_0\, E_n(0)$$

where C_0 is a constant and

$E_n(0) = u_n(0) - u(0)$, hence

$$E_{n+1}(x)=E_n(x)-E_n(x)-C_0\,E_n(0)+I^{\alpha}I^{\beta}[k_1(y_n(x))-k_1(y(x))]+I^{\alpha}I^{\gamma}[k_2(y_n(x))-k_2(y(x)]$$

with $y_n(0) = y(0)$, then $E_n(0) = 0$. Therefore:

$$\begin{aligned}E_{n+1}(x) &= I^{\alpha}I^{\beta}[k_1(y_n(x))-k_1(y(x))]+I^{\alpha}I^{\gamma}[k_2(y_n(x))-k_2(y(x)]\\ &= I^{\alpha+\beta}[k_1(y_n(x))-k_1(y(x))]+I^{\alpha+\gamma}[k_2(y_n(x))-k_2(y(x)]\\ &= I^{\alpha_1}[k_1(y_n(x))-k_1(y(x))]+I^{\alpha_2}[k_2(y_n(x))-k_2(y(x)]\end{aligned}$$

where $\alpha_1 = \alpha + \beta$ and $\alpha_2 = \alpha + \gamma$, then

$$E_{n+1}(x)=\frac{1}{\Gamma(\alpha_1)}\int_0^x (x-s)^{\alpha_1-1}[k_1(y_n(s))-k_1(y(s))]ds+\frac{1}{\Gamma(\alpha_2)}\int_0^x (x-s)^{\alpha_2-1}+[k_2(y_n(s))-k_2(y(s))]ds$$

$$\begin{aligned}&=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x (x-s)^{\alpha-1}\{\Gamma(\alpha_2)(x-s)^{\beta}k_1(y_n(s))-\Gamma(\alpha_2)(x-s)^{\beta}k_1(y(s))\\ &\quad+\Gamma(\alpha_1)(x-s)^{\gamma}k_2(y_n(s))-\Gamma(\alpha_1)(x-s)^{\gamma}k_2(y(s))\}ds\\ &=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x (x-s)^{\alpha-1}[\{\Gamma(\alpha_2)(x-s)^{\beta}k_1(y_n(s))+\Gamma(\alpha_1)(x-s)^{\gamma}k_2(y_n(s))\}\\ &\quad-\{\Gamma(\alpha_2)(x-s)^{\beta}k_1(y(s))+\Gamma(\alpha_1)(x-s)^{\gamma}k_2(y(s))\}]ds\end{aligned}$$

Then

$$E_{n+1}(x)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x (x-s)^{\alpha-1}\left[k(y_n(s))-k(y(s))\right]ds$$

Now, taking the maximum norm of the two sides of E_{n+1}, will give

$$\begin{aligned}\left\|E_{n+1}(x)\right\|_\infty &=\left\|\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x (x-s)^{\alpha-1}[k(y_n(s))-k(y(s))]\,ds\right\|_\infty\\ &\le\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x \|x-s\|_\infty^{\alpha-1}\,\|k(y_n(s))-k(y(s))\|_\infty\,ds\\ &\le\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x \max_{s\in[0,x]}|x-s|^{\alpha-1}L\|y_n(s)-y(s)\|_\infty\,ds\\ &=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x x^{\alpha-1}L\|E_n(s)\|_\infty\,ds\end{aligned}$$

Hence: $\left\|E_{n+1}(x)\right\|_\infty \le \dfrac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\alpha-1}\displaystyle\int_0^x \|E_n(s)\|_\infty\,ds,\ \forall\, n = 0, 1, ...$

Now, if n = 0, then:

$$\begin{aligned}\left\|E_{n+1}(x)\right\|_\infty &\le\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\alpha-1}\int_0^x \|E_0(s)\|_\infty\,ds\\ &\le\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\alpha-1}\int_0^x \max_{s\in[0,x]}|E_0(s)|\,ds\\ &\le\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\alpha-1}\max_{s\in[0,x]}|E_0(s)|\int_0^x ds\\ &\le\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\alpha}\max_{s\in[0,x]}|E_0(s)|\end{aligned}$$

Also, for n = 1, we have:

$$\left\|E_{n+1}(x)\right\|_{\infty} \le \frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha-1} \int_0^x \| E_1(s) \|_{\infty} \, ds$$

$$\le \frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha-1} \int_0^x \left[\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} s^{\alpha} \max_{s\in[0,x]} | E_0(s) | \right] ds$$

$$\le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^2 x^{\alpha-1} \max_{s\in[0,x]} | E_0(s) | \int_0^x s^{\alpha} \, ds$$

$$\le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^2 \frac{x^{2\alpha}}{\alpha+1} \max_{s\in[0,x]} | E_0(s) |$$

Similarly, for n = 2, then:

$$\left\|E_{n+1}(x)\right\|_{\infty} = \frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha-1} \int_0^x \| E_2(s) \|_{\infty} \, ds$$

$$\le \frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha-1} \int_0^x \left[\left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^2 \frac{s^{2\alpha}}{\alpha+1} \max_{s\in[0,x]} | E_0(s) | \right] ds$$

$$\le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^3 x^{\alpha-1} \max_{s\in[0,x]} | E_0(s) | \int_0^x \left[\frac{s^{2\alpha}}{\alpha+1} \right] ds$$

$$\le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^3 x^{\alpha-1} \max_{s\in[0,x]} | E_0(s) | \frac{x^{2\alpha+1}}{(\alpha+1)(2\alpha+1)}$$

$$\le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^3 \frac{x^{3\alpha}}{(\alpha+1)(2\alpha+1)} \max_{s\in[0,x]} | E_0(s) |$$

$\vdots$

$$\left\|E_{n+1}(x)\right\| \le \left(\frac{L}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right)^n \frac{x^{n\alpha}}{(\alpha+1)(2\alpha+1)\cdots((n-1)\alpha+1)} \max_{s\in[0,x]} | E_0(s) |$$

and since $L < [\Gamma(\alpha + \beta)\, \Gamma(\alpha + \gamma)]$, so as $n \longrightarrow \infty$, we have $\|E_n(x)\|_{\infty} \longrightarrow 0$, i.e., $y_n \longrightarrow y$. ■

6. Illustrative Examples:

In this section, we shall present two integro-differential equations of fractional order, linear and nonlinear, which will be solved using the above method of solution.

<u>**Example (1):**</u>

Consider the following linear integro-differential equations of fractional order:

$$D^{0.5}y(x) = \frac{6}{\Gamma(3.5)} x^{2.5} - \frac{6}{\Gamma(4.5)} x^{3.5} - \frac{6}{\Gamma(4.75)} x^{3.75} + I^{0.5}y(x) + I^{0.75}y(x) \qquad (19)$$

where u(0) = 0, $x \in [0,1]$.

Then by eq. (16), we have the following variational iteration formula related to eq. (30):

$$y_{n+1}(x)=y_n(x)-I^{0.5}\{D^{0.5}y_n(x)-\frac{6}{\Gamma(3.5)}x^{2.5}+\frac{6}{\Gamma(4.5)}x^{3.5}+\frac{6}{\Gamma(4.75)}x^{3.75}-I^{0.5}k\left(y_n(x)\right)-I^{0.75}k_2(y_n(x)\}$$

and consider the initial approximation

$y_0(x) = y(0) = 0$, then:

$$y_1(x)=y_0(x)-I^{0.5}\{{}^{C}D^{0.5}y_0(x)-\frac{6}{\Gamma(3.5)}x^{2.5}+\frac{6}{\Gamma(4.5)}x^{3.5}+\frac{6}{\Gamma(4.75)}x^{3.75}-I^{0.5}y(x)-I^{0.75}y(x)\}$$

$$=-I^{0.5C}D^{0.5}y_0(x)+I^{0.5}\left(\frac{6}{\Gamma(3.5)}x^{2.5}\right)-I^{0.5}\left(\frac{6}{\Gamma(4.5)}x^{3.5}\right)-I^{0.5}\left(\frac{6}{\Gamma(4.75)}x^{3.75}\right)$$

$$+I^{0.5}I^{0.5}y_0(x)+I^{0.5}I^{0.75}y_0(x)$$

and upon using the properties of fractional differentiation and integration, we get:

$$y_1(x)=y_0^{(0)}(0^+)\frac{x^0}{0!}+\frac{6}{\Gamma(3.5)}\frac{\Gamma(2.5+1)}{\Gamma(2.5+1+0.5)}x^{2.5+0.5}-\frac{6}{\Gamma(4.5)}\frac{\Gamma(3.75+1)}{\Gamma(3.5+1+0.5)}x^{3.5+0.5}$$

$$-\frac{6}{\Gamma(4.75)}\frac{\Gamma(3.75+1)}{\Gamma(3.75+1+0.5)}x^{3.75+0.5}+\int_0^x y_0(0)dt+I^{1.25}y_0(0)$$

$$=\frac{6}{\Gamma(4)}x^3-\frac{6}{\Gamma(5)}x^4-\frac{6}{\Gamma(5.25)}x^{4.25}+\int_0^x y_0(0)dt+I^{1.25}y_0(0)$$

$$y_1(x)=x^3-\frac{1}{4}x^4-\frac{6}{\Gamma(5.25)}x^{4.25}$$

Similarly:

$$y_2(x)=x^3-\frac{1}{20}x^5-\frac{12}{\Gamma(6.25)}x^{5.25}-\frac{6}{\Gamma(6.5)}x^{5.5}$$

$$y_3(x)=x^3-\frac{1}{120}x^6-\frac{18}{\Gamma(7.25)}x^{6.25}-\frac{18}{\Gamma(7.5)}x^{6.5}-\frac{6}{\Gamma(7.75)}x^{6.75}$$

$$y_4(x)=x^3-\frac{1}{840}x^7-\frac{24}{\Gamma(8.25)}x^{7.25}-\frac{36}{\Gamma(8.5)}x^{7.5}-\frac{24}{\Gamma(8.75)}x^{7.75}-\frac{6}{\Gamma(9)}x^8$$

and so on.

Therefore, using the mathematical induction, one may conclude that the approximate solution converges to the exact solution $y(x) = x^3$ as $n \longrightarrow \infty$.

The comparisons between the exact and approximate results are given in table (1).

Table 1- The absolute error between the exact and approximate solutions of example (1).

x	**$\lvert y(x) - y_1(x)\rvert$**	**$\lvert y(x) - y_2(x)\rvert$**	**$\lvert y(x) - y_3(x)\rvert$**	**$\lvert y(x) - y_4(x)\rvert$**	**$\lvert y(x) - y_5(x)\rvert$**
0	0	0	0	0	0
0.1	3.46×10^{-5}	9.31×10^{-7}	2.084×10^{-8}	3.81×10^{-9}	1.49×10^{-9}
0.2	5.82×10^{-4}	3.29×10^{-5}	1.513×10^{-6}	3.022×10^{-8}	2.808×10^{-8}
0.3	3.05×10^{-3}	2.66×10^{-4}	1.89×10^{-5}	9.714×10^{-7}	1.147×10^{-7}
0.4	9.87×10^{-3}	1.18×10^{-3}	1.138×10^{-4}	9.046×10^{-6}	3.788×10^{-7}
0.5	2.46×10^{-2}	3.73×10^{-3}	4.594×10^{-4}	4.733×10^{-5}	4.00×10^{-6}
0.6	5.18×10^{-2}	9.59×10^{-3}	1.439×10^{-3}	1.81×10^{-4}	1.95×10^{-5}
0.7	9.74×10^{-2}	2.13×10^{-2}	3.78×10^{-3}	5.628×10^{-4}	7.183×10^{-5}
0.8	0.168	4.26×10^{-2}	8.737×10^{-3}	1.503×10^{-3}	2.221×10^{-4}
0.9	0.27	7.9×10^{-2}	1.8×10^{-2}	3.58×10^{-3}	6.013×10^{-4}
1	0.42	0.136	3.03×10^{-2}	7.78×10^{-3}	1.476×10^{-3}

Example (2):
Consider the following nonlinear integro-differential equations of fractional order:

$$ {}^{c}D^{0.75}y(x)=\frac{1}{\Gamma(1.25)}x^{0.25}-\frac{1}{\Gamma(2.5)}x^{1.5}-\frac{2}{\Gamma(3.25)}x^{2.25}+I^{0.25}[y(x)]^{2}+I^{0.5}y(x) \qquad (20)$$

where $u(0) = 0$, $x\in[0,1]$.
Then by eq. (16), we have the following variational iteration formula related to eq. (31):

$$y_{n+1}(x)=y_n(x)-I^{0.75}\{D^{0.75}y_n(x)-\frac{1}{\Gamma(1.25)}x^{0.25}+\frac{1}{\Gamma(2.5)}x^{1.5}+\frac{2}{\Gamma(3.25)}x^{2.25}-I^{0.25}\left[y_n(x)\right]^2-I^{0.75}y_n(x)\}$$

and consider the initial approximation
$y_0(x) = y(0) = 0$, then:

$$y_1(x)=y_0(x)-I^{0.75}\{{}^{C}D^{0.75}y_0(x)-\frac{1}{\Gamma(1.25)}x^{0.25}+\frac{1}{\Gamma(2.5)}x^{1.5}+\frac{2}{\Gamma(3.25)}x^{2.25}-I^{0.25}[y_0(x)]^2-I^{0.5}y_0(x)\}$$

$$=-I^{0.75C}D^{0.75}y_0(x)+I^{0.75}\left(\frac{1}{\Gamma(1.25)}x^{0.25}\right)-I^{0.75}\left(\frac{1}{\Gamma(2.5)}x^{1.5}\right)-I^{0.75}\left(\frac{2}{\Gamma(3.25)}x^{2.25}\right)$$

$$+I^{0.75}I^{0.25}[y_0(x)]^2+I^{0.75}I^{0.5}y_0(x)$$

and upon using the properties of fractional differentiation and integration, we get:

$$y_1(x)=y_0^{(0)}(0^+)\frac{x^0}{0!}+\frac{1}{\Gamma(1.25)}\frac{\Gamma(0.25+1)}{\Gamma(0.25+1+0.75)}x^{0.25+0.75}-\frac{1}{\Gamma(2.5)}\frac{\Gamma(1.5+1)}{\Gamma(1.5+1+0.75)}x^{1.5+0.75}$$

$$-\frac{2}{\Gamma(3.25)}\frac{\Gamma(2.25+1)}{\Gamma(2.25+1+0.75)}x^{2.25+0.75}+\int_0^x y_0(0)dt+I^{1.25}y_0(0)$$

$$=x-\frac{1}{\Gamma(3.25)}x^{2.25}-\frac{1}{3}x^3+\int_0^x u_0(0)dt+I^{1.25}u_0(0)$$

$$y_1(x)=x-\frac{1}{\Gamma(3.25)}x^{2.25}-\frac{1}{3}x^3$$

Similarly:

$$y_2(x)=x-\frac{1}{\Gamma(4.5)}x^{3.5}-0.242x^{4.25}-\frac{2}{15}x^5+0.028x^{5.5}+0.042x^{6.25}+\frac{1}{63}x^7$$

and so on.
Therefore, using the mathematical induction, one may conclude that the approximate solution converges to the exact solution $y(x) = x$ as $n\longrightarrow\infty$.

The comparisons between the exact and approximate results are given in table (2).

Table 2- The absolute error between the exact and approximate solutions of example (2).

X	**$\lvert y(x) - y_1(x)\rvert$**	**$\lvert y(x) - y_2(x)\rvert$**	**$\lvert y(x) - y_3(x)\rvert$**	**$\lvert y(x) - y_4(x)\rvert$**	**$\lvert y(x) - y_5(x)\rvert$**
0	0	0	0	0	0
0.1	2.5392×10^{-3}	4.201×10^{-5}	4.46×10^{-7}	2.385×10^{-8}	2.781×10^{-8}
0.2	1.31598×10^{-2}	6.032×10^{-4}	1.891×10^{-5}	3.179×10^{-7}	1.249×10^{-7}
0.3	3.51282×10^{-2}	2.9827×10^{-3}	1.771×10^{-4}	7.785×10^{-6}	1.314×10^{-7}
0.4	7.12472×10^{-2}	9.4279×10^{-3}	8.96×10^{-4}	6.325×10^{-5}	3.321×10^{-6}
0.5	0.1241315	2.31896×10^{-2}	3.1546×10^{-3}	3.264×10^{-4}	2.675×10^{-5}
0.6	0.1962873	4.84998×10^{-2}	8.9228×10^{-3}	1.259×10^{-3}	1.418×10^{-4}
0.7	0.2901488	9.04642×10^{-2}	2.14904×10^{-2}	3.958×10^{-3}	5.84×10^{-4}
0.8	0.4080985	0.1548582	4.58322×10^{-2}	1.06588×10^{-2}	1.9948×10^{-3}
0.9	0.5524796	0.2478203	8.87521×10^{-2}	2.5381×10^{-2}	5.8779×10^{-3}
1	0.7256045	0.3754321	0.1587195	5.45793×10^{-2}	1.53363×10^{-2}

7. Conclusions:

1. The VIM, as it is known, is a very accurate method, which gives the exact solution in a few steps, but in some cases it requires more calculations which will add some difficulties to the problem under consideration.
2. The VIM derived for fractional integro-differential equations which was derived in section four may be considered as the generalization to the VIM obtained by other researchers for solving other type of equations, with certain values of α, β, and γ.

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