Effect of Magnetichydrodynamic on unsteady flow and heat transfer upon stretching sheet with non – uniform heat

Tagreed kasim Ibrahim*, Ahmed M. Abdul Hadi
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract:-

In this paper we study the effect of magnetichydrodynamic upon the boundary layer flow and heat transfer on a permeable unsteady stretching sheet with non – uniform heat source / sink. It found that the momentum and energy equations are controlled by many different dimensionless parameters such as prandtle number p_r, unsteadiness parameter A, constant pressure S_o, coefficient of the space dependent A^*, the temperature dependent B^*, and the MHD parameter M. The analytic solutions are obtained by using suitable similarity transformations and homotopy analysis method (HAM).

Furthermore, we analysis the effects of all dimensionless number, there are mentioned above, upon the velocity distribution and hest transfer distribution. This study is done through plotting (36) graph by using the Mathematica package.

Keywords: MHD, unsteady flow, stretching sheet , non- uniform heat.

تأثير الحقل المغناطيسي على جريان لامستقر وعلى انتقال الحرارة في صفيحة مطاطية ذات مصدر حراري غير منتظم

تغريد قاسم ابراهيم*، احمد مولود عبد الهادي
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق.

الخلاصة:

في هذا البحث درسنا تأثير الحقل المغناطيسي على جريان لامستقر على الطبقة المتاخمة ، وانتقال الحرارة في صحيفة مطاطية بوجود مصدر حراري غير منتظم . لقد وجد ان معادلات الحركة ومعادلة الطاقة يتحكم بها عدد معين من الاعداد اللابعدية مثل عدد براندل ، معامل اللاستقرارية ،الضغط المستمر ، معامل الفضاء المعتمد، درجة الحرارة المعتمد ،معامل الحقل المغناطيسي . لقد تم حل هذه المعادلات باستخدام طريقة الهوموتوبي التحليلية .

بالاضافة الى ذلك ، تمت دراسة تأثير كل من الاعداد اللابعدية المذكورة اعلاه على كل من حقل توز يع السرعة وحقل توزيع الحرارة . وهذه الدراسة تمت من خلال رسم حوالي (36) بيان باستخدام البرنامج الجاهز ماثيماتيكا .

1-Introduction

Fluid is that state of matter, which is capable of changing shape and is capable of flowing. Fluids may be classified as real "viscous" and ideal "perfect" according to whether the fluid is capable of

*Email:ta12_28@yahoo.com

exerting shearing stress or not. Real fluid is called Newtonian if the relation between stress and rate of strain is linear, otherwise is called non-Newtonian fluid.

Within the past fifty years, many problems dealing with the flow of
Newtonian and non-Newtonian fluids through porous channels have been studied by engineers and mathematicians.

The analysis of such flows find important applications in engineering practice, particularly in chemical industries; investigations of such fluids are desirable. A number of industrially important fluids including molten plastics, polymers, pulps, foods and fossil fuels, which may saturate in underground beds, display non-Newtonian behavior.

The magnetohydrodynamic (MHD) phenomenon is characterized by an interaction between the hydrodynamic and boundary layer and the electromagnetic field. The study of MHD flow in a channel also has applications in many devices like MHD power generators, MHD pumps, accelerators, etc.

Investigations of boundary layer flow and heat transfer are important due to its applications in industries and many manufacturing process. Grane L.J. [1] investigated the flow due to stretching sheet with linear surface velocity and obtained the similarity solution to the problem.

Vajravelu and Roper [2] studied the flow and heat transfer in a second grade fluid over a stretching sheet with viscous dissipation and internal heat generation or absorption. Hayat., and Sajid., [3] a second grade fluid has been analyzed for two heating process, namely PST-case and PHF –case. The series solutions were obtained through homotopy analysis method.

Sarma [4] presented an analytical solution for heat transfer in a steady laminar flow of an incompressible viscoelastic fluid past a semi – infinite stretching sheet with power – law surface temperature, including the effects of viscous dissipation and internal heat generation or absorption .The flow of a power – law fluid with variable thermal conductivity and non – uniform heat source.

These studied were confined to the steady state conditions. Since the stretching sheet may varies with time, the aspect of unsteady stretching sheet becomes interesting in practical problems. Anderssona [5] explored the heat transfer in liquid film over unsteady stretching sheet, and obtained numerical solutions. Wang [6] gave the analytical solutions of this problem. Mukhopadhyay [7, 8] extended it by assuming the viscosity and thermal diffusivity are linear functions of temperature and studied mix convection boundary layer flow of an incompressible viscous liquid through porous medium along permeable surface, the thermal radiation effect to heat transfer was also considered .The effect of non- uniform heat source of laminar flow over unsteady stretching sheet was explored by Tsai [9].

Sajid etal.[10] investigated the analytic solution for MHD flow and heat transfer in a third – order fluid over a stretching sheet.

Anuar etal [11] studied heat transfer over an unsteady stretching permeable surface with prescribed wall temperature .

Since the steady of heat source /sink effect on heat transfer in some cases, in the present paper we study the unsteady boundary flow and heat transfer on a permeable stretching sheet with non-uniform source.

The homotopy analysis method (HAM) [12] is employed to give analytic solution. (HAM) was first proposed by Liao [12 -14].

Many workers have used this method to solve various non-linear problems successfully, as list of the key references in the vast literature concerning these fields we refer the resent papers by Hang etal, , Ali, Memmood, and Ziabakhsh Domairry [15-17].

Liancun Zheng, etal[18], study the effect of magnetic hydrodynamic upon the boundary layer flow and heat transfer on a permeable unsteady stretching sheet with non – uniform heat source / sink .

Van Gorder and Vajravelu [19] discussed the selection of initial approximation, auxiliary linear operator, auxiliary function, and convergence control parameter in the application of the homotopy analysis method.

The effect Magnetohydrodynamic stuied by Cowling. and Gramer [20,21] The symmetric steady problem, the fluid having different normal velocities at the channel walls the effect of the various parameters of interest for the velocity and temperature are pointed out.

2. Basic ideas of HAM:-.

The homotopy analysis method (HAM) is a technique can be used to solve non-linear partial differential equation. Now we write a systematic description for general nonlinear problems and proposed by Liao [12-14] .We consider one nonlinear equation governed by:-

$$\mathrm{N}[u(r,t)] = 0 \qquad (1)$$

where N is a nonlinear operator, $u(r,t)$ is an unknown function, and r and t denoted spatial and temporal independent variables, respectively. Let $u_o(r,t)$ (an initial guess of exact solution $u(r,t)$ ، $h \neq 0$ an auxiliary parameter, $H(r,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator. then, using $q \in [0,1]$ as an embedding parameter, we construct such a homotopy

$$(1-q)\{L[\phi(r,t;q) - u_o(r,t)]\} = qh\, H(r,t) N\, [\phi(r,t;q)] \qquad (2)$$

where $\phi(r,t;q)$ is the solution which depends not only upon $u_o(r,t)$, L, $H(r,t)$ and h but also upon $q \in [0,1]$, Liao [12,13] expanded $\phi(r,t;q)$ in Taylor series about the embedding parameter

$$\phi(r,t;q) = u_o(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) q^m \qquad (3)$$

where

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}\bigg|_{q=o} \qquad (4)$$

The convergence of the series (3) depends upon the auxiliary parameter h . If it is convergent at $q = 1$, one has

$$u(r,t) = u_o(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) \qquad (5)$$

Dividing the zero- order deformation equation m - times with respect to q and then dividing them by $m!$ and then finally setting $q = 0$, we obtain the following m th –order deformation problem:

$$L[u_m(r,t) - X_m u_{m-1}(r,t)] = hH(r,t) R_m(\vec{u}_{m-1}, r, t) \qquad (6)$$

In which

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m \geq 2 \end{cases} \qquad (7)$$

$$R_m(\vec{u}_{m-1}, r, t) = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} N\left[\sum u_n(r,t) q^n \right] \right\}\bigg|_{q=0} \qquad (8)$$

where

$$\vec{u}_n = \{u_o(r,t), u_1(r,t), u_2, (r,t) \ldots, u_n(r,t)\}$$

There are many different ways to get the higher order deformation equation. However, according to the fundamental theorem in calculus, the term $u_m(r,t)$ in the series (3) is unique. Note that the HAM contains an auxiliary parameter h, which provides us with a simple way to control and adjust the series solution (5).

3. Boundary layer governing equations:-

Consider the unsteady, two dimensional, incompressible viscous flow on a stretching permeable surface in a quiescent fluid, and the sheet is stretching with a velocity $U_w = \frac{ax}{1-ct}$ in the positive x direction .Here $a > 0$,$b > 0$ and $t < \frac{1}{c}$.The sheet surface temperature $T_w = T_\infty + \frac{bx}{1-ct}$ varies with coordinate x and time t . The boundary layer governing equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2 u}{\rho} \tag{10}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right] \tag{11}$$

The associated boundary layer conditions are:

$$u=U_w(x,t),\ \ v=v_w=-(\frac{va}{1-ct})^{\frac{1}{2}}S_o,\ \ T=T_w(x,t)\ \text{ at }\ y=0 \tag{12}$$

$$u\to 0\ ,\ T\to T_\infty \qquad \text{at}\quad y\to\infty \tag{13}$$

where u, v are the velocity, k is the thermal conductivity , c_p is the specific heat, ρ is the density, S_o constant pressure, $v=\frac{\mu}{\rho}$ is the kinematic coefficient of viscosity, B is the applied magnetic field vector, and T is the temperature.

The internal heat source\sink of q is chosen as:

$$q=(\frac{kU_w}{xv})[A^*(T_w-T_\infty)f^{/}+B^*(T-T_\infty)], \tag{14}$$

where A^* coefficient of the space dependent, and B^* the temperature dependent.

we introduce the following dimensionless parameters and similarity transformations:

$$\psi=(\frac{va}{1-ct})^{1/2}xf(\eta)\ ,\eta=(\frac{a}{v(1-ct)})^{\frac{1}{2}}y\ ,\theta(\eta)=\frac{T-T\infty}{T_w-T_\infty} \tag{15}$$

where $\psi(x,y,z,t)$ is a stream function satisfies the mass equation

Using (15), we have:

$$f'''+M\ f'+ff''-f'^2-A(f'+\frac{\eta}{2}f'')=0 \tag{16}$$

$$\theta''+P_r\left[f\theta'-f'\theta-\frac{A}{2}(2\theta+\eta\theta')\right]+A^*f'+B^*\theta=0 \tag{17}$$

and the boundary conditions are:

$$f(0)=S_o\ \ ,\ f'(0)=1\ \ ,\ \theta(0)\ =1\ , \tag{18}$$

$$f'(\infty)=0\ ,\ \theta(\infty)\ =0\ \ , \tag{19}$$

where $P_r=\frac{v}{\frac{k}{\rho_{c\,p}}}$ is Prandtle number, and $A=\frac{c}{a}$ is the unsteadiness parameter.

4. Solution of the governing equation:-

In this section, we attempt to obtain analytical solutions for the imposed problem.

4.1. Basic procedure:-

For the HAM solving procedure, we solved the problem by:

1- Zero-order deformation equations:-

Solving Eqs. (16)-(19) using HAM [13-14] from the boundary conditions (18) and (19), it is obvious to choose:

$$f_o(\eta)=S_o+1-\exp(-\eta)\,, \tag{20}$$

$$\theta_o(\eta)=\exp(-\eta)\,, \tag{21}$$

as the initial approximations of $f(\eta)$ and $\theta(\eta)$, respectively and to choose:

$$L_f = \frac{\partial^3}{\partial\eta^3} + \frac{\partial^2}{\partial\eta^2}, \quad (22)$$

$$L_\theta = \frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}, \quad (23)$$

as the auxiliary linear operators, which have the following properties:

$$L_f(c_1 + c_2\eta + c_3\exp(-\eta) = 0, \quad L_\theta(c_4 + c_5\exp(-\eta) = 0 \quad (24)$$

where c_1 , c_2 , c_3, c_4 and c_5 are arbitrary constants.

Based on (16) and (17), this paper is led to define the following properties:

$$N[\phi(\eta,q)] = \phi''' + M\phi' + \phi\phi'' - \phi'^2 - A(\phi' + \frac{\eta}{2}\phi''), \quad (25)$$

$$N[\Theta(\eta;q)] = \theta'' + p_r\left[f\theta' - f'\theta - \frac{A}{2}(2\theta + \eta\theta')\right] + A^* f' + B^*\theta \quad (26)$$

Let h denote the non –zero an auxiliary parameter. Then construct the zero- order deformation equation:

$$(1-q)L_f[\phi(\eta;q) - f_o(\eta)] = qhH_f(\eta)N_f[\phi(\eta;q)], \quad (27)$$

$$(1-q)L_\theta[\Theta(\eta;q) - \theta_o(\eta)] = qhH_\theta(\eta)N_\theta[\Theta(\eta;q)], \quad (28)$$

subject to the boundary conditions:

$$\phi(0;q) = S_o, \quad \left.\frac{\partial\phi(\eta;q)}{\partial\eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial\phi(\eta;q)}{\partial\eta}\right|_{\eta=\infty} = 0, \quad (29)$$

$$\Theta(0;q) = 1, \quad \Theta(\infty;q) = 0, \quad (30)$$

where $q \in [0,1]$ is an embedding parameter. When $q = 0$, it is straightforward that:

$$\phi(\eta,0) = f_o(\eta), \quad \Theta(\eta;0) = \theta_o(\eta), \quad (31)$$

When $q = 1$, the zero- order deformation equations (27)-(30) are equivalent to the original equations (16)-(19), so that we have:

$$\phi(\eta,q) = f(\eta), \quad \Theta(\eta;q) = \theta(\eta) \quad (32)$$

when q increases from 0 to 1, $\phi(\eta,q)$ and $\Theta(\eta,q)$ vary from the initial guess $f_o(\eta)$ and $\theta_o(\eta)$ to the solution $f(\eta)$ and $\theta(\eta)$ of the problem, respectively. So expanding $\phi(\eta,q)$ and $\Theta(\eta,q)$ in Taylor's series about the embedding parameter q we have:

$$\phi(\eta,q) = f_o(\eta) + \sum_{m=1}^{\infty} f_m(\eta)q^m, \quad (33)$$

$$\Theta(\eta,q) = \Theta_o(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)q^m, \quad (34)$$

where :

$$f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\phi(\eta;q)}{\partial q^m}\right|_{q=0}, \quad (35)$$

$$\theta_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\Theta(\eta;q)}{\partial q^m}\right|_{q=0}, \quad (36)$$

The convergence of the series (33) and (34) depends upon h. We choose h in such a way that the series (33) and (34) is convergence at $q = 1$; then due to Eq. (31) and (32) we have:

$$f(\eta) = f_o(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (37)$$

$$\theta(\eta) = \theta_o(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad (38)$$

2. Higher-order deformation equations:-

Differentiating m times the zero- order deformation equations (27) and (28) with respect to q and then dividing it by $m!$ and finally setting $q=0$, we get the following mth- order deformation equations:

$$L[f_m(\eta)-\chi_m f_{m-1}(\eta)] = hHR_m(\vec{f}_{m-1}), \quad (39)$$

$$L[\theta_m(\eta)-\chi_m \theta_{m-1}(\eta)] = hHS_m(\vec{\theta}_{m-1}), \quad (40)$$

subject to the boundary conditions:

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad (41)$$

$$\theta_m(0) = \theta_m(\infty) = 0, \quad (42)$$

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m \ge 2 \end{cases}, \quad (43)$$

and

$$R_m(\eta) = f'''_{m-1}(\eta) + \sum_{k=0}^{m-1} f_k(\eta) f''_{m-k-1}(\eta) - \sum_{k=0}^{m-1} f'_k(\eta) f'_{m-k-1}(\eta) - (A+M) f'_{m-1}(\eta) - \frac{\eta A}{2} f''_{m-1} \quad (44)$$

$$S_m(\eta) = \theta''_{m-1}(\eta) + p_r \begin{bmatrix} \sum_{k=0}^{m-1} f_k(\eta)\theta'_{m-k-1}(\eta) - \sum_{k=0}^{m-1} f'_k(\eta)\theta_{m-1-k}(\eta) - \\ \frac{A}{2}(2\theta_{m-1}(\eta) + \eta\theta'_{m-1}(\eta)) \end{bmatrix} + A^* f'_{m-1}(\eta) + B^* \theta_{m-1}(\eta) \quad (45)$$

According to initial approximation and the auxiliary linear operators, we set:

$$H_f(\eta) = \exp(-\eta) \quad , \quad H_\theta(\eta) = \exp(-\eta) \quad (46)$$

The first order deformation equations:

$$L_f[f_1(\eta)] = hH_f(\eta)R_1(\eta), \quad (47)$$

$$L_\theta[\theta_1(\eta)] = hH_\theta(\eta)S_1(\eta), \quad (48)$$

and the boundary conditions:

$$f_1(0) = f'_1(0) = f'_1(\infty) = 0, \quad (49)$$

$$\theta_1(0) = \theta_1(\infty) = 0, \quad (50)$$

For the solution of the high- order problem, we use the symbolic computation software MATHEMATICA up to first few order of approximation. We found the solution up to seventh- order and fourth –order approximation of momentum and energy equation respectively.

5- General solution:-

5-1 General solution of momentum equation:-

It is found that the general solution of momentum equation is given by:

$$f_m(\eta) = \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i \, b_{m,k}^i \, \eta^i \, e^{-k\eta} \quad (51)$$

where $b_{m,k}^i$ are the coefficients of $f_m(\eta)$ for $m \ge 1$

Now , we try to find the above coefficient:

1. The initial approximation $f_o(\eta)$ defined by Eq. (20) has the same structure as Eq.(51).
2. If we assume that the first $(m-1)$ solutions $f_k(\eta)$ $(k = 0,1,2,\ldots,m-1)$ have the same structure as (51), then we want to prove that $f_m(\eta)$ has the same structure as (51) to prove this, we have from Eq.(51):

$$f_m' = \sum_{k=0}^{2m} \sum_{i=1}^{2(2m-k)} i\,\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-1}\, e^{-k\eta} + \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta}(-k)$$

$$= \sum_{k=0}^{2m} \sum_{i=0}^{4m-2k-1} (i+1)\,\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i\, e^{-k\eta} + \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta}(-k)$$

$$= \sum_{k=0}^{2m} \left[\sum_{i=0}^{2(2m-k)} (i+1)\,\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i - k \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i \right] e^{-k\eta}$$

$$f_m' = \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} a_{m,k}^i\, \eta^i\, e^{-k\eta} \qquad (52)$$

where

$$a_{m,k}^i = (i+1)\,\lambda_{m,k}^{i+1} b_{m,k}^{i+1} - k\,\lambda_{m,k}^i\, b_{m,k}^i$$

$$f_m'' = \sum_{k=0}^{2m} \sum_{i=2}^{2(2m-k)} i(i-1)\,\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-2}\, e^{-k\eta} + \sum_{k=0}^{2m} \sum_{i=1}^{2(2m-k)} i\,\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-1}\, e^{-k\eta}(-k)$$

$$+ \sum_{k=0}^{2m} \sum_{i=1}^{2(2m-k)} i\,\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-1}(-k)\, e^{-k\eta} + \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} i\,\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-1}\, e^{-k\eta} k^2$$

$$= \sum_{k=0}^{2m} \sum_{i=0}^{4m-2k-2} (i+2)(i+1)\,\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2}\, \eta^i\, e^{-k\eta} - 2k \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} (i+1)\,\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i\, e^{-k\eta}$$

$$+ k^2 \sum_{k=0}^{2m} \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta}$$

$$f_m'' = \sum_{k=1}^{2m} \sum_{i=0}^{2(2m-k)} (i+2)(i+1)\,\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2}\, \eta^i\, e^{-k\eta} - 2k \sum_{k=1}^{2m} \sum_{i=0}^{2(2m-k)} (i+1)\,\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i\, e^{-k\eta}$$

$$+ k^2 \sum_{k=1}^{2m} \sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta}$$

$$f_m'' = \sum_{k=1}^{2m} e^{-k\eta} \left[\sum_{i=0}^{2(2m-k)} (i+2)(i+1)\,\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2} - 2k\,(i+1)\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1} + k^2\, \lambda_{m,k}^i\, b_{m,k}^i \right] \eta^i$$

$$f_m'' = \sum_{k=1}^{2m} e^{-k\eta} \sum_{i=0}^{2(2m-k)} c_{m,k}^i\, \eta^i \qquad (53)$$

where

$$c_{mk}^i = (i+1)(i+2)\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2} - 2k(i+1)\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1} - k^2 \lambda_{m,k}^i b_{m,k}^i$$

Similarly

$$f_m''' = \sum_{k=1}^{2m} e^{-k\eta} \sum_{i=0}^{2(2m-k)} d_{m,k}^i\, \eta^i \qquad (54)$$

where

$$d_{mk}^i = (i+1)\lambda_{m,k}^{i+1}\; c_{m,k}^{i+1} - k\lambda_{m,k}^i\; c_{m,k}^i$$

Now, from Eqs(51) and (53) , we have:

$$f_k(\eta)\, f_{m-1-k}''(\eta) = \sum_{J=0}^{2k} e^{-J\eta} \sum_{i=0}^{2(2k-J)} \lambda_{k,J}^i\, b_{k,J}^i\, \eta^i \sum_{r=1}^{2(m-1-k)} e^{-r\eta} \sum_{s=0}^{4(m-1-k)-2r} c_{m-1-k,r}^s \eta^s$$

$$= \sum_{J=0}^{2k} \sum_{r=1}^{2(m-1-k)} \exp[(-J+r)\eta] \sum_{i=o}^{2(2k-J)} \lambda_{k,J}^i b_{k,J}^i \sum_{s=0}^{4(m-1-k)-2r} c_{m-1-k,r}^s \eta^{s+i}$$

$$= \sum_{J=0}^{2k} \sum_{r=1}^{2(m-1-k)} \exp[(-J+r)\eta] \sum_{i=o}^{2(2k-J)} \lambda_{k,J}^{i} b_{k,J}^{i} \sum_{s=0}^{4(m-1-k)-2r} c_{m-1-k,r}^{s} \eta^{s+i}$$

$$= \sum_{n=1}^{2(m-1)} \exp[-(n)\eta] \sum_{J=\max[1,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=o}^{2(2k-J)} \sum_{s=0}^{4(m-1-k)-2n+2J} \lambda_{k,J}^{i} b_{k,J}^{i} c_{m-1-k,r}^{s} \eta^{s+i}$$

$$= \sum_{n=1}^{2(m-1)} \exp[-(n)\eta] \sum_{J=\max[1,n-2(m-1-k)]}^{\min[n,2k]} \sum_{q=o}^{4(m-1)-2n} \eta^{q} \sum_{i=\max[o,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]}$$

$$= \sum_{n=1}^{2(m-1)} \exp[-(n)\eta] \sum_{q=o}^{4(m-1)-2n} \eta^{q} \sum_{J=\max[1,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=\max[o,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]} c_{m-1-k,n-J}^{q-i} \lambda_{k,J}^{i} b_{k,J}^{i}$$

which further gives

$$\sum_{k=0}^{m-1} f_k(\eta)\, f''_{m-1-k}(\eta) = \exp(-\eta) \sum_{q=0}^{4(m-1)-2n} \eta^{q} S_{m,1}^{q} + \sum_{n=2}^{2(m-1)} \exp(-n\eta) \sum_{q=0}^{4(m-1)-2n} \eta^{q} S_{m,n}^{q} \qquad (55)$$

where

$$S_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{J=\max[1,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=\max[o,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]} c_{m-1-k,n-J}^{q-i} \lambda_{k,J}^{i} b_{k,J}^{i}$$

Now, from Eq (52), we have:

$$f'_k(\eta)\, f'_{m-1-k}(\eta) = \sum_{J=0}^{2k} e^{-J\eta} \sum_{i=0}^{2(2k-J)} a_{k,J}^{i} \eta^{i} \sum_{r=0}^{2(m-1-k)} e^{-r\eta} \sum_{s=0}^{4(m-1-k)-2r} a_{m-1-k,r}^{s} \eta^{s}$$

$$= \sum_{J=0}^{2k} \sum_{r=0}^{2(m-1-k)} \exp[(-J+r)\eta] \sum_{i=o}^{2(2k-J)} \sum_{s=0}^{4(m-1-k)-2r} a_{k,J}^{i} a_{m-1-k,r}^{s} \eta^{s+i}$$

$$= \sum_{n=0}^{2(m-1)} \exp[-(n)\eta] \sum_{J=\max[0,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=0}^{2(2k-J)} \sum_{s=0}^{4(m-1-k)-2n+2J} a_{k,J}^{i} a_{m-1-k,r}^{s} \eta^{s+i}$$

$$= \sum_{n=0}^{2(m-1)} \exp[-(n)\eta] \sum_{J=\max[0,n-2(m-1-k)]}^{\min[n,2k]} \sum_{q=o}^{4(m-1)-2n} \eta^{q} \sum_{i=\max[0,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]} a_{k,j}^{i} a_{m-1-k,n-J}^{q-i}$$

$$= \sum_{n=0}^{2(m-1)} \exp[-(n)\eta] \sum_{q=o}^{4(m-1)-2n} \eta^{q} \sum_{J=\max[0,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=\max[o,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]}$$

which further gives

$$\sum_{k=0}^{m-1} f'_k(\eta)\, f'_{m-1-k}(\eta) = \sum_{n=0}^{2(m-1)} \exp(-\eta) \sum_{q=0}^{4(m-1)-2n} \eta^{q} \Delta_{m,1}^{q} + \sum_{n=1}^{2(m-1)} \exp(-n\eta) \sum_{q=0}^{4(m-1)-2n} \eta^{q} \Delta_{m,n}^{q} \qquad (56)$$

where

$$\Delta_{m,n}^{q} = \sum_{k=0}^{m-1} \sum_{J=\max[0,n-2(m-1-k)]}^{\min[n,2k]} \sum_{i=\max[o,q-4(m-1-k)-2n+2J]}^{\min[q,2(2k-J)]} a_{m-1-k,n-J}^{q-i} a_{k,J}^{i}$$

substituting Eqs.(52),(53),(54),(55) and (56) into Eq(44), we have:

$$R_m(\eta)=\sum_{k=1}^{2(m-1)} e^{-k\eta}\sum_{i=0}^{4(m-1)-2k} d_{m,k}^{i}\ \eta^{i}+\exp(-\eta)\sum_{q=0}^{4(m-1)-2n}\eta^{q}S_{m,1}^{q}$$

$$+\sum_{n=2}^{2(m-1)}\exp(-n\eta)\sum_{q=0}^{4(m-1)-2n}\eta^{q}S_{m,n}^{q}-\sum_{n=0}^{2(m-1)}\exp(-\eta)\sum_{q=0}^{4(m-1)-2n}\eta^{q}\Delta_{m,1}^{q}$$

$$+\sum_{n=1}^{2(m-1)}\exp(-n\eta)\sum_{q=0}^{4(m-1)-2n}\eta^{q}\Delta_{m,n}^{q}-(A+m)\sum_{k=0}^{2(m-1)}\sum_{i=0}^{4(m-1)-2k}a_{m-1,k}^{i}\,\eta^{i}\,e^{-k\eta}$$

$$-\frac{\eta A}{2}\sum_{k=1}^{2(m-1)}e^{-k\eta}\sum_{i=0}^{4(m-1)-2k}c_{m-1,k}^{i}\ \eta^{i}$$

$$R_m(\eta)=$$

$$\exp(-\eta)\sum_{q=0}^{2(2m-1)}T_{m,1}^{q}\,\eta^{q}+\sum_{n=2}^{2(m-1)}e^{-n\eta}\sum_{q=0}^{4(m-1)-2n}T_{m,n}^{q}\,\eta^{q} \quad (57)$$

where

$$T_{m,1}^{q}=(d_{m-1,1}^{q}+S_{m,1}^{q}+\Delta_{m,1}^{q}-(A+M)\,a_{m-1,1}^{q}-\eta\frac{A}{2}c_{m-1,1}^{q})$$

$$T_{m,m+1}^{q}=(S_{m,n}^{q}+\Delta_{m,n}^{q})$$

$$T_{m,n}^{q}=\begin{cases}(d_{m-1,n}^{q}+S_{m,n}^{q}+\Delta_{m,n}^{q}-(A+M)\,a_{m-1,n}^{q}-\eta\frac{A}{2}c_{m-1,n}^{q}) & 0\le q\le 2(2m-n)\\ (S_{m,n}^{q}+\Delta_{m,n}^{q}) & 0\le q\le 4(m-1)-2n\\ 0 & otherwise\end{cases}$$

The above equation is the right hand side of the following equation

$$L[f_m(\eta)-\chi_m f_{m-1}(\eta))]=hHR_m(\overrightarrow{f_{m-1}}) \quad (58)$$

By solving the last equation, we found that the general solution is given by:

$$L(\phi_m-\chi_m\phi_{m-1})=$$

$$\exp(-\eta)\sum_{q=0}^{2(2m-1)}T_{m,1}^{q}\,\eta^{q}+\sum_{n=2}^{2(m-1)}e^{-n\eta}\sum_{q=0}^{4(m-1)-2n}T_{m,n}^{q}\,\eta^{q} \quad (59)$$

$$L(\phi_m-\chi_m\phi_{m-1})=\exp(-\eta)\left[\sum_{q=0}^{2(m-1)}T_{m,1}^{q}\mu_{1,0}^{q}+\sum_{k=1}^{2m-1}\eta^{k}(\sum_{q=k-1}^{2(m-1)}T_{m,1}^{q}\mu_{1,k}^{q})\right]$$

$$-\sum_{n=2}^{2(2m-1)}\exp(-n\eta)\left[\sum_{k=0}^{4(m-1)-2n}\eta^{k}(\sum_{q=k}^{4(m-1)-2n}T_{m,n}^{q}\mu_{n,k}^{q})\right] \quad (60)$$

$$+c_1^{m}\exp(-\eta)+c_2^{m}+\eta c_3^{m}$$

where c_1^m , c_2^m and c_3^m are integral constants. Using the boundary conditions;

$$f_m(0)=f_m'(0)=f_m'(\infty)=0$$

$$0=\sum_{q=0}^{2(m-1)}T_{m,1}^{q}\mu_{1,0}^{q}-\sum_{n=2}^{2(2m-1)}\left[T_{m,n}^{0}\ \mu_{n,0}^{0})\right]+c_1^{m}+c_2^{m} \quad (61)$$

$$L(\phi'_m-\chi_m\phi'_{m-1}) = -\exp(-\eta)\left(\sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q\right) + \exp(-\eta)\left(\sum_{k=1}^{2m-1} k\eta^{k-1} \sum_{q=k-1}^{2(m-1)} T_{m,1}^q \mu_{1,k}^q\right)$$
$$-\exp(-\eta)\left(\sum_{k=1}^{2m-1}\eta^k \sum_{q=k-1}^{2(m-1)} T_{m,1}^q \mu_{1,k}^q\right) - \sum_{n=2}^{2(2m-1)} -n\exp(-n\eta)\left(\sum_{k=0}^{4(m-1)}\eta^k \sum_{q=k}^{4(m-1)-2n} T_{m,n}^q \mu_{n,k}^q\right) \quad (62)$$

we have;

$$c_3^m = 0$$

$$c_1^m = \sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q - \sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,1}^q - \sum_{n=2}^{2(2m-1)} n \left(\sum_{q=0}^{4(m-1)-2n} T_{m,n}^q \mu_{n,0}^q\right)$$
$$+ \sum_{n=2}^{2(2m-1)} \sum_{q=1}^{4(m-1)-2n} T_{m,n}^q \mu_{n,1}^q$$

$$c_2^m = -\sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q + \sum_{n=2}^{2(2m-1)} \left[T_{m,n}^0\ \mu_{n,0}^q)\right] - \sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q + \sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,1}^q +$$
$$\sum_{n=2}^{2(2m-1)} n \left(\sum_{q=0}^{4(m-1)-2n} T_{m,n}^q \mu_{n,0}^q\right) - \sum_{n=2}^{2(2m-1)} \sum_{q=1}^{4(m-1)-2n} T_{m,n}^q \mu_{n,1}^q$$

Then from above equations, one can get the coefficient of Eq.(51),

We obtain in fact the following explicit, totally analytic solution of the present flow.

$$f_m(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M\to\infty} \sum_{m=0}^{M}\left(\sum_{k=0}^{2m}\sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta}\right) \quad (63)$$

5-2 General solution of energy equation:-

To find the general solution of energy equation by the same procedure of momentum equation and is given by:

$$\theta_m(\eta) = \sum_{k=0}^{2m}\sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta} \quad (64)$$

then:

$$L(\Theta_m-\chi_m\Theta_{m-1}) = \exp(-\eta)\left[\sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q + \sum_{k=1}^{2m-1}\eta^k \left(\sum_{q=k-1}^{2(m-1)} T_{m,1}^q \mu_{1,k}^q\right)\right]$$
$$-\sum_{n=2}^{2(2m-1)} \exp(-n\eta)\left[\sum_{k=0}^{4(m-1)-2n}\eta^k\left(\sum_{q=k}^{4(m-1)-2n} T_{m,n}^q \mu_{n,k}^q\right)\right] + c_3^m \exp(-\eta) + \eta\ c_4^m \quad (65)$$

where c_3^m and c_4^m are integral constants. Using the boundary conditions

$$\theta_m(0) = \theta_m(\infty) = 0$$

we have:

$$c_4^m = 0$$

$$0 = \sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q - \sum_{n=2}^{2(2m-1)} T_{m,n}^0 \mu_{n,0}^0 + c_3^m$$

$$c_3^m = -\sum_{q=0}^{2(m-1)} T_{m,1}^q \mu_{1,0}^q + \sum_{n=2}^{2(2m-1)} T_{m,n}^0 \mu_{n,0}^0$$

Then from above equations, one can get the coefficient of Eq.(64).

We obtain in fact the following explicit, totally analytic solution of heat transfer.

$$\theta_m(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta) = \lim_{M\to\infty} \sum_{m=0}^{M}\sum_{k=0}^{2m}\sum_{i=0}^{2(2m-k)} \lambda_{m,k}^i\, b_{m,k}^i\, \eta^i\, e^{-k\eta} \quad (66)$$

(6) Convergence of The Solution:-

It is noticed that the explicit, analytical expressions (51) and (64) contain auxiliary parameter h. As pointed out by Lio [14], the convergence region and rate of approximations given by homotopy analysis method are strongly dependent upon h. Figure-1) and (2) portray the h-curves of the velocity and temperature profiles respectively. The range for admissible values of h for the velocity is $-2 \le h \le 2$ and for temperature it is $-2 \le h \le 2$. We see that the series given by Eqs. (63) and (66) converges in the whole region of η when $h = -1.2$ and $h = -0.2$. This value of h lie in the admissible range of h.

To choose the h-curves of the velocity and temperature profiles from the table (1), (2) and (3) respectively:

Table 1- h-curves of the velocity up to seventh order

h-**Curves**	**The First Derivative**	**The Second Derivative**	**Order**
-0.2	0.796882	-0.952637	Seventh
-0.4	0.777811	-0.979492	Seventh
-0.6	0.759932	-0.962679	Seventh
-0.8	0.74476	-0.935881	Seventh
-1.0	0.733229	-0.912466	Seventh
-1.2	0.723477	-0.891657	Seventh
-1.4	0.709828	-0.861689	Seventh
-1.6	0.682698	-0.800156	Seventh
-1.8	0.629138	0.671754	Seventh
-2.0	0.533739	-0.423627	Seventh

Table 2- h-curves of the velocity up to fifth order

h-**Curves**	**The First Derivative**	**The Second Derivative**	**Order**
-0.2	0.803498	-0.930565	Fifth
-0.4	0.781714	-0.965664	Fifth
-0.6	0.760405	-0.960604	Fifth
-0.8	0.743596	-0.94013	Fifth
-1.0	0.732114	-0.917289	Fifth
-1.2	0.723386	-0.893555	Fifth
-1.4	0.71124	-0.858964	Fifth
-1.6	0.685707	-0.79224	Fifth
-1.8	0.632818	-0.660928	Fifth
-2.0	0.534407	-0.421523	Fifth

Table 3- h-curves of the temperature up to fourth order

h-**Curves**	**The First Derivative**	**The Second Derivative**	**Order**
-0.2	-0.821265	0.822237	Fourth
-0.4	-0.826408	0.828123	Fourth
-0.6	-0.83047	0.82944	Fourth
-0.8	-0.829762	0.81924	Fourth
-1.0	-0.820594	0.790576	Fourth
-1.2	-0.799276	0.736498	Fourth
-1.4	-0.76212	0.650059	Fourth
-1.6	-0.705436	0.52431	Fourth
-1.8	-0.625533	0.352303	Fourth
-2.0	-0.518723	0.12709	Fourth

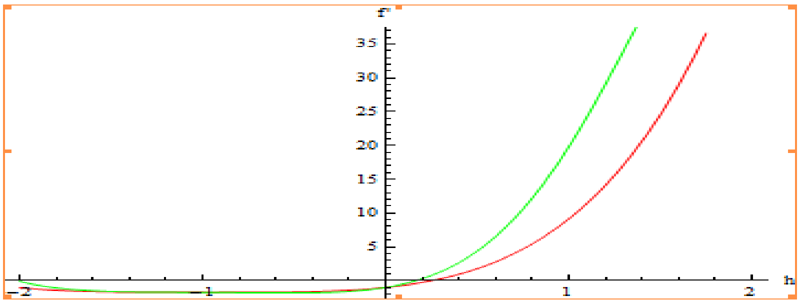

Figure 1- h -curve for velocity at seventh- order approximation

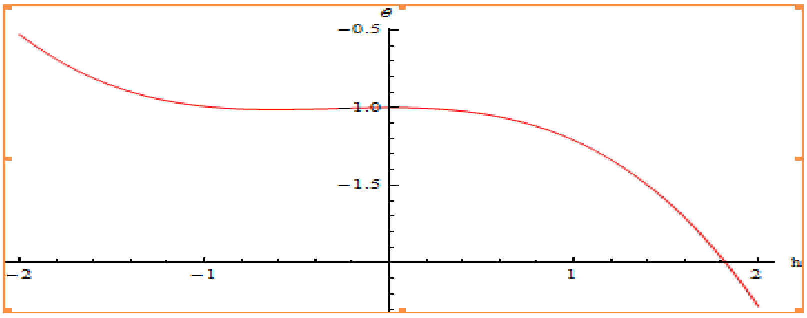

Figure 2- h -curve for temperature at fourth- order approximation

7- Results and discussion:-

Utilizing the analytical solutions, calculations are performed to investigate the effect of MHD parameter M , constant pressure S_o , the unsteadiness parameter A, coefficient of the space dependent A^* , the temperature dependent B^* , prandtle number p_r .The following results are made of**:**

7-1 Velocity distribution:

1- As MHD parameter " M " increases, there is an increasing in the velocity rang. See figure-3.

2- As constant pressure " S_o " increases, there is an increasing in the velocity rang. See figure-4 .

3- As unsteadiness parameter " A" increases, there is an decreasing in the velocity rang. See figure-5

7-2 Temperature distribution

1- As MHD parameter " M " increases, there is an decreasing in the temperature rang See figure-6.

2- As coefficient of the space dependent " A^* " decreases, there is an decreasing in the temperature rang. See figure-7.

3- As temperature dependent " B^* " dencreases, there is an increasing in the temperature rang. See figure-8 .

4- As prandtle number " p_r " increases, there is an decreasing in the temperature rang. See figure-9 .

5- As constant pressure " S_o " increases, there is an increasing in the temperature rang. See figure-10.

6- As unsteadiness parameter " A" increases, there is an increasing in the temperature rang. See figure-11.

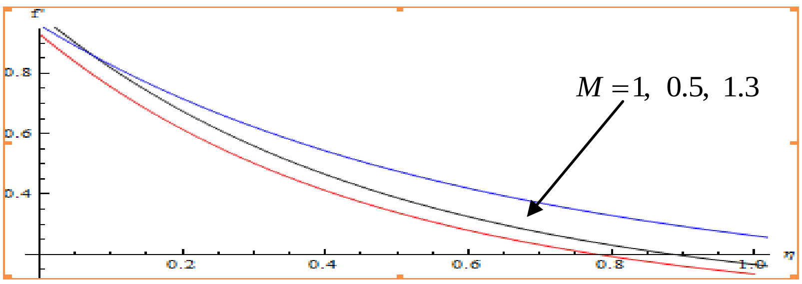

Figure 3- Velocity distribution f, for $S_o = 1.5$, $A = 0.8$

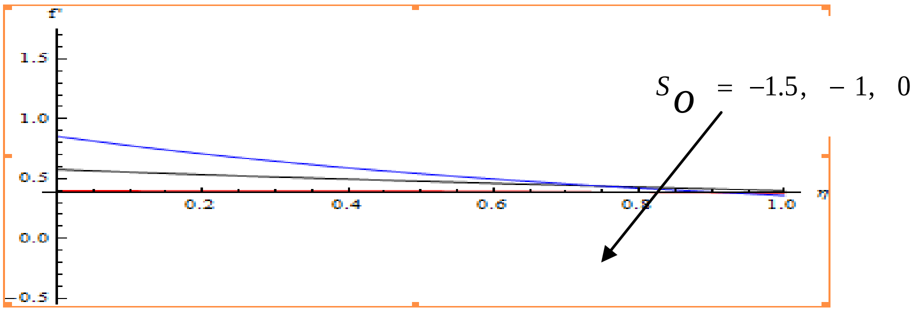

Figure 4- Velocity distribution f, for $M = 0.5$, $A = 0.8$

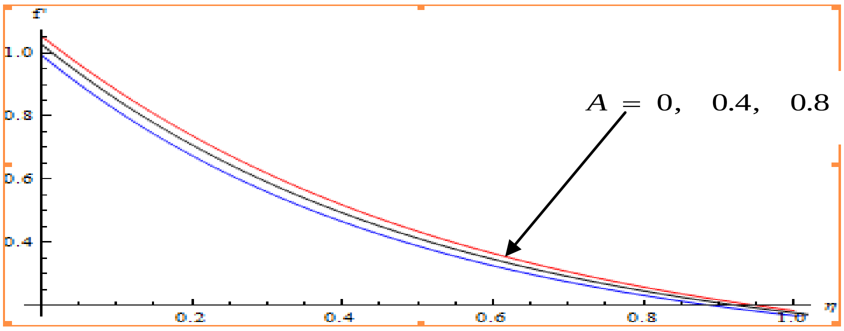

Figure 5- Velocity distribution f, for $M = 0{,}5$, $S_o = 1.5$

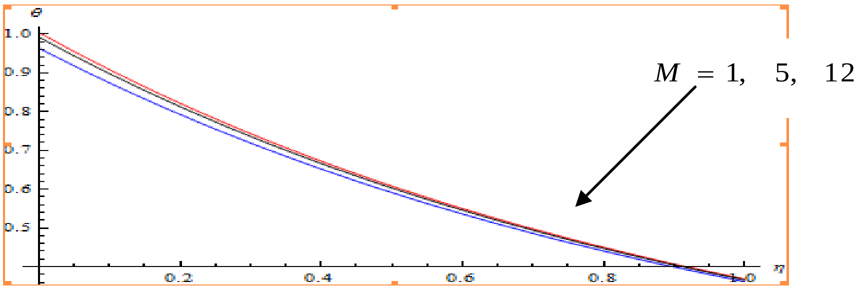

Figure 6- Temperature distribution θ, for $A = 0.8$, $P_r = 0.7$, $S_o = 1.5$, $A^* = 0.05$, $B^* = 0.6$

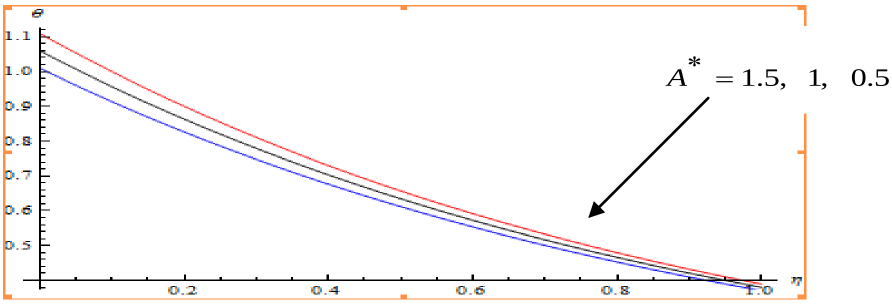

Figure 7- Temperature distribution θ, for $A = 0.8$, $P_r = 0.7$, $S_o = 1.5$, $M = 12$, $B^* = 0.05$

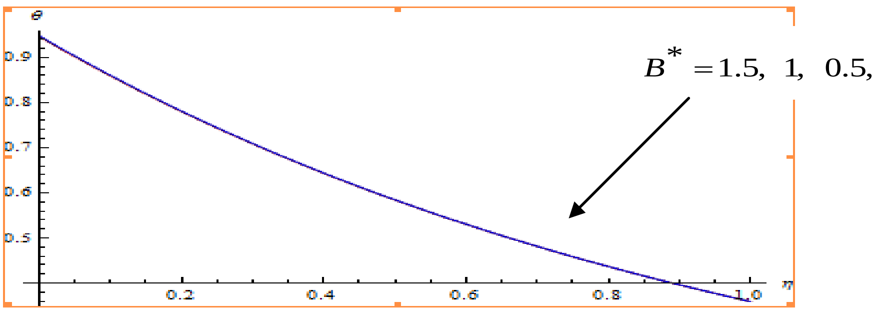

Figure 8- Temperature distribution θ, for $A = 0.8$, $P_r = 0.7$, $S_o = 1.5$, $M = 15$, $A^* = 0.05$

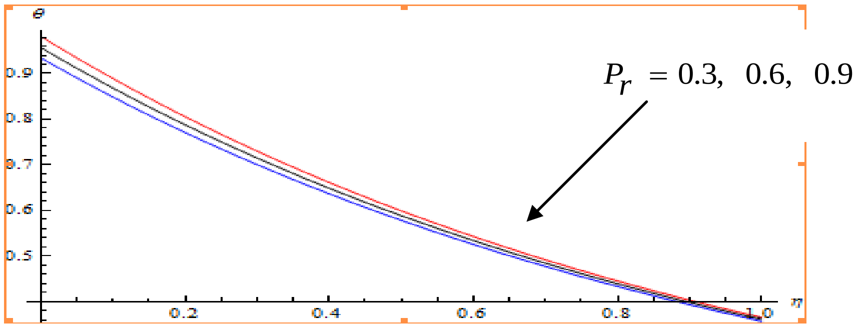

Figure 9- Temperature distribution θ, for $A=0.8$, $B^*=0.05$, $S_o=1.5$, $M=15$, $A^*=0.05$

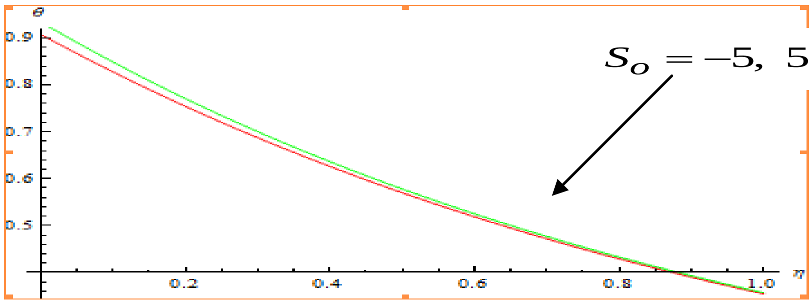

Figure 10- Temperature distribution θ, for $A=0.8$, $B^*=0.05$, $P_r=0.7$, $M=15$, $A^*=0.05$

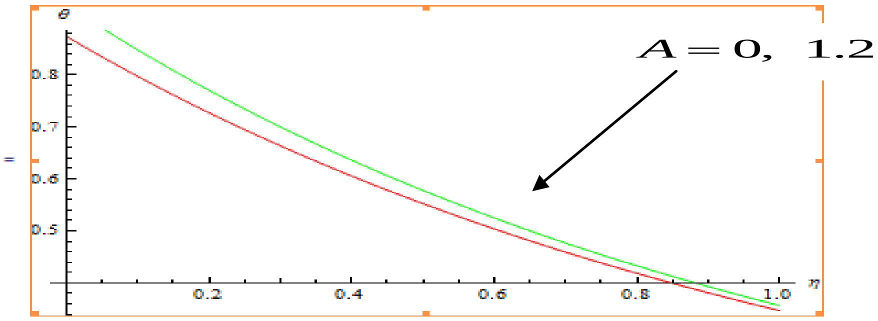

Figure 11- Temperature distribution θ, for $S_o=1.5$, $B^*=0.05$, $P_r=1$, $M=20$, $A^*=0.05$

Reference:-

1. Grane LJ. **1970**, Flow Past Stretching Plate, *ZAngew Math Phys*, Vol.*21*,pp645-7.
2. Vajravelu k. and Roper T. **1999**, Flow and Heat Transfer in a Second Grade Fluid over a Stretching Sheet, *Int. J Non-linear Mech*,Vol.*34*, pp:103-6.
3. Hayat T., and sajid M. **2007**, Analytic Solution for Axisymmetric Flow and Heat Transfer of a Second Grade Fluid Past a Stretching Sheet, *Int J Heat Mass Transfer*,Vol.*50*, pp:75-84.
4. Sarma Ms. **1998**, Heat Transfer in a Viscoelastic Fluid over a Stretching Sheet, *J. Math Apple*, Vol.*222*, pp:268-75.
5. Anderssona HI, Aarseth JB, and Dandapat BS. **2000**, Heat Transfer in Liquid Film over Unsteady Stretching Surface, *Int. J. Heat Mass Transfer*, Vol.*43*, pp:69-74.
6. Wang C. **2006**, Analytic Solutions for Liquid Film over Unsteady Stretching Surface, *Heat Mass Transfer*,Vol.*42*, pp:759-66.
7. Mukhopadhyay S. **2009** , Unsteady Boundary Layer Flow and Heat Transfer Past a Porous Stretching Sheet in Presence of Variable Viscosity and Thermal Diffusivity, *Int. J. Heat Mass Transfer*, Vol. *52*,pp5213-5217.
8. Mukhopadhyay S. **2009**, Effect of Thermal Radiation on Unsteady Mixed Convection Flow and Heat Transfer over a Porous Stretching Surface in Porous Medium, *Int. J. Heat Mass Transfer*, Vol. *52*,pp3261-5.
9. Tsai R, Huang KH, and Huang JS. **2008**, Flow and Heat Transfer over an Unsteady Stretching Surface with Non-Uniform Heat Source, *Int. Commun Heat Mass Transfer*, Vol.*35*, pp1340-3.

10. Sajid M. , Hayat T. and Asghar S. **2007**, Non – Similar Analytic Solution for MHD Flow and Heat Transfer in a Third – Order Fluid over a Stretching Sheet, *Int. J Heat Mass Transfer*, Vol.*50*, pp 1723-1736 .

11. Anuar Ishak, Roslinda Nazar, Ioan Pop **2009**, Heat Transfer over an Unsteady Stretching Permeable Surface with Prescribed Wall Temperature, *Nonlinear Analysis :Real World Applications*, Vol.*10*, pp2909-2913.

12. Liao S. J. **2003**, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, Chapman & Hall, Boca Raton.

13. **.** Liao S. J. **1999**, A Uniformly Valid Analytic Solution of Two-Dimensional Viscous Flow over A Semi-Infinite Flat Plate , *J. Fluid Mech*. Vol.*385*, pp101-128 .

14. Liao S. J. **2003,** An Analytic Approximate Technique for free Oscillations of Positively Damped Systems with Algebraically Decaying Amplitude, *Int. J. Non-linear Mech*.Vol.*38*, pp173-1183.

15. Hang X , Lio SJ , and Pop I**. 2007**, Series Solutions of Unsteady Three – Dimensional MHD Flow and Heat Transfer in The Boundary Layer over an Impulsively Stretching Plate, *Eur J Mech B/fluids*, Vol.*26*, pp15-27.

16. Ali A., and Memmood A. **2008**, Homotopy Analysis of Unsteady Boundary Layer Flow Adjacent to Permeable Stretching Surface in a Porous Medium, *Common Nonlinear Sci Nmer Simui*, Vol.*13*, pp340-9.

17. Ziabakhsh Z. and Domairry G. **2009**, Analytic Solution of Natural Convection Flow of a Non – Newtonian Fluid between two Vertical Fat Plates Using Homotopy Analysis Method, .*Commum Nonlinear Sci. Number*, Vol. *14*, pp1868-80.

18. Liancun Zheng, Lijuan, Wang and Xinxin Zhang **2011**, Study the Effect of Magnetic Hydrodynamic upon The Boundary Layer Flow and Heat Transfer on a Permeable Unsteady Stretching Sheet with Non – Uniform Heat Source / Sink*], Commum Nonlinear Sci Number Simulat* ,Vol.*16*, pp731-740.

19. Van Gorder R. A .and Vajravelu K. **2009**, On The Selection of Auxiliary Functions, Operators, and Convergence Control Parameters in The Application of The Homotopy Analysis Method to Nonlinear Differential Equations: A General Approach, *Commun. Nonlinear Sci. Numer. Simul.*, Vol.*14*, pp4078-4089.

20. Cowling T. G. **1957**, *Magnetohydrodynamics*, Interscience Publishers, Inc. New York.

21. Gramer K. R. and Pai-Shih-1, **1973**, *Magnetofluiddynamics for Engineers and Applied Physics*, McGraw-Hill Book Company, New York.