Choosing Between Trimmed L-moment and L-moment Estimators of Extreme Value Distribution (Type- I)

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Abstract
Trimmed Linear moments (TL-moments) are natural generalization of L-moments that do not require the mean of the underlying distribution to exist. It is known that the sample TL-moments is unbiased estimators to corresponding population TL-moment. Since different choices for the amount of trimming give different values of the estimators it is important to choose the estimator that has minimum mean squares error than others. Therefore, we derive an optimal choice for the amount of trimming from known distributions based on the minimum errors between the estimators. Moreover, we study simulation-based approach to choose an optimal amount of trimming and maximum like hood method by computing the estimators and mean squares error for range of trimming and choose the one which has minimum mean squares error .

Keywords: Trimmed Linear moments, Extreme Value theory, Gumbel distribution, probability weighted moments.

اختيار أفضل طريقة تقدير من بين طريقة العزوم الخطية وطريقة العزوم الخطية المعممة لتقدير معالم توزيع القيمة المتطرفة (النوع الأول) .

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الخلاصة:
تعتبر طريقة العزوم الخطية حالة خاصة من طريقة العزوم المعممة والتي لا تتطلب إيجاد المعدل للتوزيع .وان العزوم الخطية المعمِمة للعينة تكون مقدر غير متحيز للعزوم الخطية المعممة للمجتمع. الطريقة المعممة تفترض حذف اصغر عنصر بالعينة أو اكبر عنصر أو كلاهما ا وبتالي يتم الحصول على مقدرات مختلفة لتوزيع القيمة المتطرفة (توزيع كمبل) وتمت المقارنة بين المقدرات باستخدام متوسط مربعات الخطأ .وتم إجراء تجارب المحاكاة للمقارنة بين المقدرات بطريقة العزوم الاعتيادية وطريقة العزوم المعممة وأيضا مقا رنتها مع طريقة الإمكان الأعظم.

Introduction

Extreme Value distributions arise as limiting distributions for maximum or minimum (extreme values) of a sample of independent and identically distributed random variables, as the sample size increases. Extreme Value theory (EVT) is the theory of modeling and measuring events which occur with very small probability [1]. This model was widely used in risk management, finance, insurance,

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economics, hydrology, material sciences, telecommunications, and many other industries dealing with extreme events. Extreme Value Distributions (EVDs) essentially involves three types of extreme value distributions, Gumbel (type I), Frechet (type II) and Weibull (type III) [2]. The Gumbel distribution, named after one of the pioneer scientists in practical applications of the Extreme Value theory (EVT), the German mathematician Emil Gumbel (1891-1966), has been extensively used in various fields including hydrology for modeling extreme events. Gumbel applied EVT on real world problems in engineering and in meteorological phenomena such as annual flood flows [1].Gumbel or type I extreme value distribution had cumulative distribution as:

$$F_x(X) = \exp\left[-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right] \quad , -\infty < x < \infty , \quad -\infty < \mu < \infty , \ \sigma \succ 0 \quad \ldots\ldots.(1)$$

Where μ, σ location and scale parameters, respectively.

And the probability density function is:

$$f_x(x) = \tfrac{1}{\sigma} e^{-(x-\mu)/\sigma} \exp[-\exp(\frac{x-\mu}{\sigma})] \quad -\infty < x < \infty , -\infty < \mu < \infty , \sigma \succ 0 \quad \ldots\ldots(2)$$

The quantile function or inverse function is:

$x(F) = \mu - \sigma \log(-\log F)$, $0 < F < 1$ ……………(3)

Hosking and Wallis applied L-moments(LMOM) estimation method to extreme value distribution and they found that it performs better than method of moments and that both methods do well in small samples compared to maximum likelihood method[3],[4].

Elamir and Seheult introduced an extension of L-moment called trimmed L-moment (TLMOM) where they trim one smallest and largest value from the conceptual sample [5] . Shabri and Zakaria applied LMOM and TLMOM of Generalized Logistic Distribution [6]. Elsayed applied trimmed L-moment Generalized extreme value distribution[7]. In this study we proposed the trimmed L-moments (TLMOM) with one smallest value were trimmed (TLMOM(1,0)) , trimmed L-moments (TLMOM) with one largest value were trimmed (TLMOM(0,1)) , trimmed L-moments (TLMOM) with one smallest and largest values were trimmed (TLMOM(1,1))from the samples of Extreme Value Distributions (EVDs).the performance of the TLMOM(1,0),TLMOM(0,1)and TLMOM(1,1) were compared with LMOM and maximum likelihood method through a simulation study .Mean square error criterion was used to compare among the estimators for different sample sizes .

Materials and methods

1. L-moment method

L-moment had been defined by Hosking [8] as a linear combinations of probability weighted moments (PWMs).He developed the theories of L-moment from the order statistics. L- moments are analogous to ordinary moments, and they can be used to summarize theoretical probability distributions and sample characteristics [4].Let $X_1, X_2, \ldots, X_r$ be a random sample of size r and let $X_{1:r}, X_{2:r}, \ldots, X_{r:r}$ .denote the corresponding order statistics[9].The rth L-moments defined by Hosking [4] as:

$$\lambda_r = r^{-1}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} E(X_{r-j:r}) \quad , r=1,2,\ldots \quad \ldots\ldots\ldots\ldots(4)$$

$E(X_{i:r})$ can by written as

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 x(F)F^{i-1}(1-F)^{r-1}dF \quad \ldots\ldots\ldots\ldots..(5)$$

Where x(F)and F are quantile function and cumulative distribution , respectively.

The first four L-moments were then defined by

$$\lambda_1 = E[X_{1:1}] \quad \ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

$$\lambda_2 = \frac{1}{2}E[X_{2:2} - X_{1:2}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7)$$

$$\lambda_3 = \frac{1}{3}E[X_{3:3} - 2X_{2:3} + X_{1:3}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(8)$$

$$\lambda_4 = \frac{1}{4} E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}] \qquad \text{………………….(9)}$$

The coefficients of the skewness and kurtosis of the probability density function. are

$$\tau_3 = \frac{\lambda_3}{\lambda_2} \quad , \quad \tau_4 = \frac{\lambda_4}{\lambda_3}$$

2. Sample L-moments

The sample of L-moment can be estimated unbiased from the sample order statistics [10]

$$l_r = \frac{1}{(r)!\binom{n}{r}} \sum_{k=0}^{r} (-1)^k \binom{r-1}{k} \sum_{i=1}^{n} \binom{i-1}{rr-1-k} \binom{n-i}{k} x_{i:n} \qquad \text{……………………..(10)}$$

3 .Trimmed L-moments

Elamir and Seheult[5] introduce an extension of L-moment called trimmed L-moment (TL-moment) where they trim one smallest and largest value from the conceptual sample. They introduced some robust modification of Eq.4 in which $E(X_{r-k:r})$ was replaced by $E(X_{r+t_1-k\ :r+t_t+t_2})$ for each r where $t_1$samollest and $t_2$largest are trimmed from the conceptual sample They denote this as $\lambda_r^{(t_1,t_2)}$:

$$\lambda_r^{(t_1,t_2)} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+t_1-k::r+t_1+t_2}) \quad ,r=1\dots\ ,t_1,t_2=0,1,\dots\ , \qquad \text{………..(11)}$$

TL-moments involves two more values t_1 and t_2(amount of trimming) need to be chosen L-moments is special case of TL-moments for t_1=t_2 =0 which can be obtain as

$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k::r}) \qquad \text{………………………………..(12)}$$

See;[4].The expected value of order statistics was:

$$E(X_{r-k::r}) = r\binom{n}{r} \int_0^1 x(F) F^{r-1} (1-F)^{n-r} \qquad \text{…………………………………(13)}$$

See;[9].TL-moments in terms of quantile function can be written as:

$$\lambda_r^{(t_1,t_2)} = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{(r+t_1+t_2)!}{(r+t_1-k)!(t_2+k)!} \int_0^1 x(F) F^{r+t_1-k-1} (1-F)^{t_2+k} \qquad \text{…………..(14)}$$

As shown by[5],[11] TL-moments is defined for very tailed distributions and eliminate the influence of the most extreme observations by giving them zero weights. For example, when t_1=t_2 = 1, $E(X_{2:3})$ is the median of sample of size 3 , which give zero weight for first and third value.

4. Sample TL-moments

We consider estimators of population TL-moments which are functions of order statistics $X_{1:n},\dots,X_{n:n}$ of a random sample X_1 , X_2 ,..., X_n of size n . defined an unbiased estimator of population TL-moments[4] as

$$l^{(t_1,t_2)}{}_{r+1} = \frac{1}{(r+1)!\binom{n}{t_{1+t_{2+r+1}}}} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{i=1}^{n} \binom{i-1}{r+t_1-k} \binom{n-i}{t_2+k} x_{i:n} \qquad \text{………………..(15)}$$

For r= 0,1, ..,t_1+t_2+r+1≤n and i= 1,2, … , n Note that l_{r+1} reflects different information for different values of r about the distribution of the sample.

5. L-moment of the Gumbel distribution

The L-moment of Gumbel distribution are obtained by substituting the Eq.3(quantile function) into Eqs.(6-9) and the rth order probability weighted moment Br to obtain :

$$B_r = \int_0^1 X(F)F^r dF = \frac{1}{(r+1)}[\mu + \sigma(\gamma + \ln(r+1))] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(16)}$$

where γ was the Euler's constant, with approximate value 0.577215. This result was obtained by using subsequently the change of variables u = −ln F and m = (r+1)u.

$$\lambda_1 = B_0 = \mu + \gamma\sigma \quad \ldots\ldots\ldots \quad (17)$$

$$\lambda_2 = 2B_1 - B_0 = \sigma \ln 2 \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

$$\lambda_3 = 6B_2 - 6B_1 + B_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots..(19)$$

$$= \mu + (1.3869)\sigma$$

$$\lambda_4 = 20B_3 - 30B_2 + 12B_1 - B_0 \quad \ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots (20)$$

$$=0.1036\sigma$$

6.Trimmed L-moments the Gumbel distribution

Based on Eq.14 ,the first two of Trimmed L-moments the Gumbel distribution
Can be derived as follows :

$$\lambda_1^{(0,0)} = \mu + 0.5771\sigma \quad , ,r=1,\ t_1=0,t_2=0 \ \ldots\ldots \quad .(21)$$

$$\lambda_1^{(1,0)} = \mu + 1.2702\sigma \quad ,r=1,\ t_1=1,t_2=0\ldots\ldots\ldots \quad .(22)$$

$$\lambda_1^{(0,1)} = \mu - 0.116\sigma \quad ,r=1,\ t_1=0,t_2=1 \ \ldots\ldots\ldots\ldots \quad .(23)$$

$$\lambda_1^{(1,1)} = \mu + 0.4592\sigma \quad ,r=1,\ t_1=1,t_2=1 \ \ldots\ldots\ldots\ldots\ldots \quad (24)$$

$$\lambda_2^{(0,0)} = 0.6931\sigma \quad ,r=0,\ t_1=0,t_2=0\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

$$\lambda_2^{(1,1)} = 0.3537\sigma \quad ,r=2,\ t_1=1,t_2=1\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

$$\lambda_2^{(1,0)} = 0.6082\sigma \quad ,r=2,\ t_1=1,t_2=0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

$$\lambda_2^{(0,1)} = 0.4314\sigma \quad ,r=2,\ t_1=0,t_2$$

=1……………………………..(28)

The parameters μ, σ of the distribution can be estimated by Trimmed L-moment as:

$$\hat{\sigma}_{(0,0)} = \frac{l_2^{(0,0)}}{0.6931}, \quad \hat{\mu}_{(0,0)} = l_1^{(0,0)} - 0.5771\hat{\sigma}_{(0,0)}$$

$$\hat{\sigma}_{(1,0)} = \frac{l_2^{(1,0)}}{0.6082}, \quad \hat{\mu}_{(1,0)} = l_1^{(1,0)} - 1.2702\hat{\sigma}_{(1,0)}$$

$$\hat{\sigma}_{(0,1)} = \frac{l_2^{(0,1)}}{0.4314}, \quad \hat{\mu}_{(0,1)} = l_1^{(0,1)} + 0.116\hat{\sigma}_{(0,1)}$$

$$\hat{\sigma}_{(1,1)} = \frac{l_2^{(1,1)}}{0.3537}, \quad \hat{\mu}_{(1,1)} = l_1^{(1,1)} - 0.4314\hat{\sigma}_{(1,1)}$$

7. Maximum likelihood method

The maximum Likelihood (ML) estimates of μ and σ as numerical solutions of the following equations

$$\hat{\mu} = \sigma(\ln n - \ln \sum_{i=1}^{n} \exp(\frac{x_i}{\sigma})) \quad \text{..............} \quad .(29)$$

and

$$\hat{\sigma} = \bar{x} - \frac{\sum_{i=1}^{n} x_i \exp(\frac{-x_i}{\sigma})}{\sum_{i=1}^{n} \exp(\frac{-x_i}{\sigma})} \quad \text{......................} \quad .(30)$$

The estimate of σ is explicitly obtained from equation (30) and the estimate of μ is then implicitly obtained from equation (29) after the substitution of the estimate of σ[12].

Results

Simulation approach

A number of simulation experiments were conducted to investigate the properties of trimming linear moment (TL-moment) estimators for type- I Extreme value distribution (EVI) Sets of 1000 random samples of sizes varying from (30 to 100)were generated from EVI distribution The location and scale parameters(μ, σ)were set (μ =2,2.5 σ =1.5,2) for each generated sample of a given size (n=30,50,100). The estimated values and mean square error (MSE)were computed ,the representation of our results of estimators of location and scale parameters were representing in Table (1)and Table (2) respectively . The mean square error of estimators of location parameters (μ =2)when the(sample size (30,50,100),scale parameter (σ =1.5,2))were presented as shown in figures(1,2,3).the mean square error of estimators of scale parameter (σ =1.5and σ =2)when (μ =2)and (n=30,50,100) were computed as shown in figures (4,5,6).

Table 1-The simulation estimation values of two parameters of Gumbel distribution for various trimming moments when (μ =2 and σ =2)

n	Methods of estimation					
	Parameters	LMOM	TLMOM (t_1=1,t_2=1)	TLMOM (t_1=1,t_2=0)	TLMOM (t_1=0,t_2=1)	Maximum like hood method
30	$\hat{\mu}$	1.9987	1.9952	1.9957	2.0233	1.9951
	$\hat{\sigma}$	2.0050	2.0054	2.0093	1.9474	1.9738
50	$\hat{\mu}$	1.9927	1.9926	1.9955	2.0066	1.9954
	$\hat{\sigma}$	1.9984	1.9996	1.9982	1.9646	2.0215
100	$\hat{\mu}$	2.0048	2.0029	2.0032	2.0108	2.0113
	$\hat{\sigma}$	2.0034	2.0022	2.0066	2.0554	1.9831

Table 2-The simulation estimation values of two parameters of Gumbel distribution for various trimming moments when (μ =2 and σ =1.5)

n	Methods of estimation					
	Parameters	LMOM	TLMOM (t_1=1,t_2=1)	TLMOM (t_1=1,t_2=0)	TLMOM (t_1=0,t_2=1)	Maximum like hood method
30	$\hat{\mu}$	2.0033	2.0003	2.0080	1.9978	2.0219
	$\hat{\sigma}$	1.5046	1.5153	1.5023	1.4650	1.4565
50	$\hat{\mu}$	1.9937	1.9925	1.9919	2.0048	1.9930
	$\hat{\sigma}$	1.5090	1.5067	1.5118	1.4819	1.5035
100	$\hat{\mu}$	1.9990	2.0007	2.0058	2.0035	2.0035
	$\hat{\sigma}$	1.4937	1.4971	1.4897	1.4847	1.5323

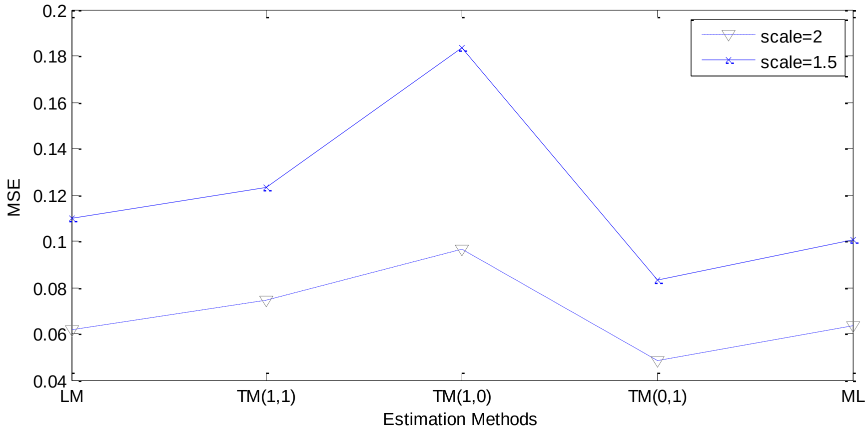

Figure 1- Mean Squares Error of scale parameter when(n=30, μ =2)

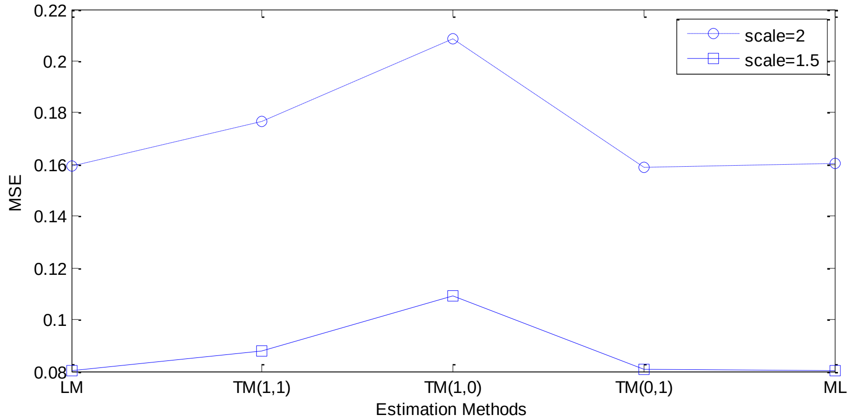

Figure 2- Mean Squares Error of location parameter when(n=30, μ =2)

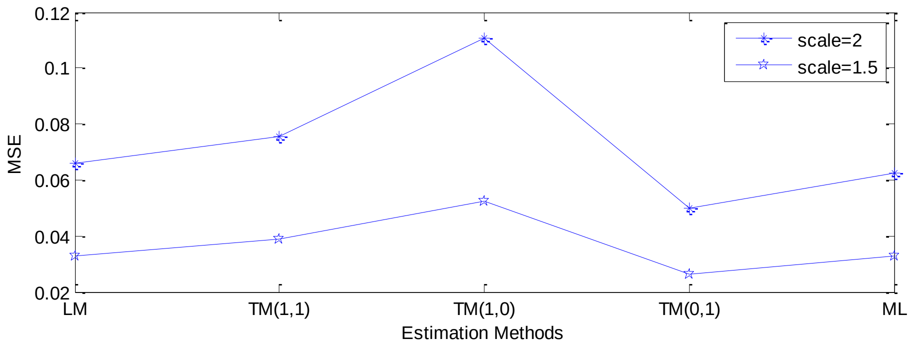

Figure 3- Mean Squares Error of scale parameter when(n=50, μ =2)

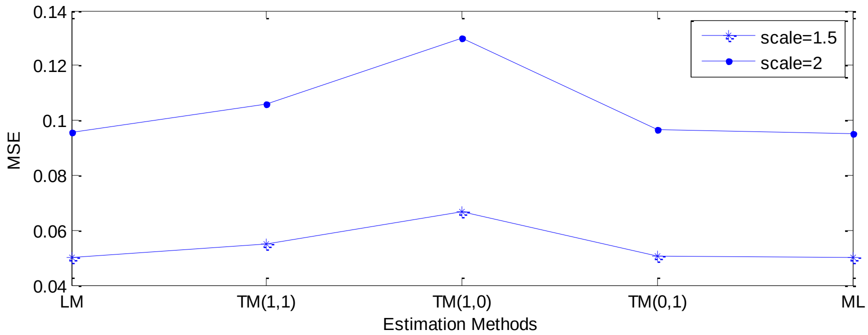

Figure 4- MSE of location parameter when(n=50, μ =2)

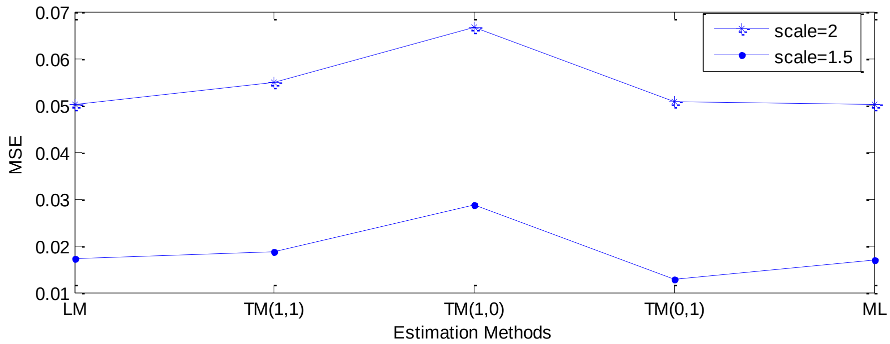

Figure 5- MSE of scale parameter when(n=100, μ =2)

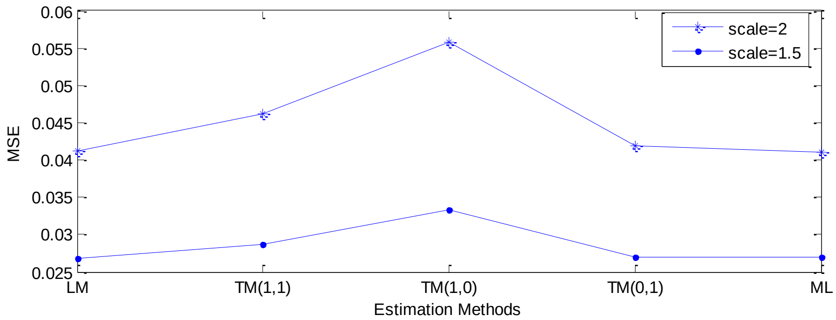

Figure 6- MSE of location parameter when(n=100, μ =2)

Discussion

The comparison was made among the method of LMOM(no data was trimmed),TLMOM(1,1)(one smallest and one largest data was trimmed), TLMOM(1,0)(one smallest data was trimmed) TLMOM(0,1)(one largest data was trimmed) and Maximum like hood method to see the performance of these methods in the sense of mean square error (MSE) for different sample size n and .the location and scale parameters were set (μ =2,2.5 , σ =1.5,2) for each generated sample of a given size (n=30,50,100) .The estimated values of the two parameters and mean square error of the estimators (MSE)were computed .The TLMOM (0,1) estimators perform much better than TLMOM (1,0) and TLMOM(1,1).It have the lowest(MES) to the other estimators for the location and scale parameters as shown in figures (1,2 ,3,4,5,6) . LMOM the second method perform much better than other methods followed by TLMOM(0,1) and the Maximum like hood method the third degree . The results show that the TLMOM was a robust version of the LMOM where TLMOM trimmed the extreme values from the sample.

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