Centralizers With Nilpotent Values

Abdul- Rahman H. Majeed, Faten Adel Shalal*
Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

ABSTRACT:
In this paper , it is shown that if R is a semiprime ring and T a centralizer of R such that $T(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer then $T = 0$.

Keywords: semiprime ring, prime ring, derivation, left (right) centralizer, centralizer, Jordan centralizer.

المتمركزات مع قيم عديمة القوى

عبد الرحمن حميد مجيد ، فاتن عادل شلال*
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة:
سنبين في هذا البحث انه لحلق ة شبه اولية R ومتمركز T من R بحيث ان $T(x)^n = 0$ لكل $(x \in R)$ حيث $(n \geq 1)$ هو عدد ثابت صحيح فأن $T=0$.

الكلمـات المفتـاحيـة : حلقة شبه اولية ، حلقة اولية ، مشتقة ، متمركز يسار (يمين) ، متمركز ، متمركز جوردن .

Introduction:

Throughout this research R will represent an associative ring. Recall that R is a prime ring if $aRb=0$ implies that $a=0$ or $b=0$ (where $a ,b \in R$) , and R is semiprime ring if $aRa=0$ implies that $a=0$ (where $a \in R$) . A ring R is 2-torsion free if $2x=0$ implies that $x=0$ (where $x \in R$) . An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in R$. An additive mapping $T: R \rightarrow R$ is called left (right) centralizer if $T(xy) = T(x)y$ $(T(xy) = x\,T(y))$ holds for all $x , y \in R$. T is called centralizer if it is both left and right centralizer . An additive mapping $T: R \rightarrow R$ is called left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $\left(T(x^2) = xT(x)\right)$ holds for all $x \in R$. Following ideas from [1], Zalar has proved in [2] that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. J. Vukman [3] shows that for a semiprime ring R with extended centroid C if $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$ holds for all $x, y \in R$ then there exists $\alpha \in C$ such that $T(x) = \alpha x$, for all $x \in R$ .Other results concerning centralizer in prime and semiprime ring can be found in [4 - 7] . In [8] it was shown that if R is

*Email: ftoonmath@ymial.com

a prime ring and d a derivation of R such that $d(x)^n = 0$ for all $x \in R$, then $d = 0$, and then extend it to the semiprime ring. Here we ask the possibility if the same result can be satisfied on R with replacing the derivation d with centralizer T. First we will prove some simple remarks which we will need them to prove our main result, for a prime ring R :

REMARK 1: If $T \neq 0$ is a centralizer of R and $aT(x) = 0$, (or, $T(x)a = 0$) for all $x \in R$ then $a = 0$.
PROOF: Since $aT(x) = 0$ for all $x \in R$, then for $r \in R$ we have
$0 = aT(rx) = arT(x)$ for all $r \in R$
Hence $aRT(x) = 0$ for all $x \in R$, by the primeness of R and using that $T \neq 0$ we get $a = 0$.
REMARK 2: If $T \neq 0$ is a centralizer of R , T does not vanish on a nonzero one sided ideal of R .
PROOF: Let I be a nonzero one sided ideal of R and suppose $T(I) = 0$.
Let $a \in I$ and $r \in R$, then
$0 = T(ar) = aT(r)$ for all $r \in R$, by Remark 1 we get $a = 0$, then $I = 0$, a contradiction, hence $T(I) \neq 0$.

REMARK 3: If $L \neq 0$ is a left ideal of R and $W = \{x \in R : Lx = 0\}$, then L/W is a prime ring.
PROOF: First one can easily show that W is a right ideal of R .
Now we will show that L/W is a prime ring . Let $(x + W)(L/W)\,(y + W) = W$, where $x, y \in R$, then $(x + W)(l + W)(y + W) = W$, where $l \in L$, this leads to $xly \in W$, hence $L(xly) = 0$ for all $l \in L$.
Let $r \in R$, hence $L(xrly) = 0$ for all $r \in R\,, l \in L$,then $(Lx)R(Ly) = 0$, by the primeness of R we get either $Lx = 0$ or $Ly = 0$. That is, either $x + W = W$ or $y + W = W$, hence L/W is a prime ring .

REMARK 4: If L is a left ideal of R and $a^m = 0$,for all $a \in L$, where m is a fixed integer, then $L = 0$.
PROOF: Suppose $L \neq 0$, then there exists $0 \neq a \in L$ such that $a^m = 0$. Let $r \in R$
$0 = (ra)^m = rara \ldots ra$, for all $r \in R$, therefore , $(ra)R(ar \ldots ra) = 0$, by the primeness of R we get either $ra = 0$ or $(ar \ldots ra) = 0$, if $ra = 0$ for all $r \in R$, $Ra = 0$, then $a = 0$, a contradiction , hence $ar \ldots ra = 0$ for all $r \in R$, hence $aR(ar \ldots ra) = 0$, again by the primeness of R we get either $a = 0$ or $(ar \ldots ra) = 0$. Continue in this way we end up with $a = 0$, a contradiction . Hence $L = 0$.

REMARK 5: If $a, b \in R$ and $(arb)^m = 0$ for all $r \in R$, where m is a fixed integer , then $ba = 0$.
PROOF: If one of a or $b = 0$ then the result holds.
Now let $a, b \neq 0$ and $(arb)^m = 0$ for all $r \in R$, then $arbarb \ldots arb = 0$ for all $r \in R$,thus $aR(barb \ldots arb) = 0$, since R is a prime ring then we have $barb \ldots arb = 0$ for all $r \in R$, hence $baR(b \ldots arb) = 0$, again since R is a prime then either $ba = 0$ or $bar \ldots arb = 0$. Continue in this way we end up with $ba = 0$.

We shall use the following notation throughout:
If S is a subset of R , then $L(S) = \{x \in R: xs = 0\,, \forall s \in S\}$, and $R(S) = \{x \in R: sx = 0, \forall s \in R\}$, clearly $L(S)$ is a left ideal and $R(S)$ is a right ideal.

In what follows R will be a prime ring and T a centralizer of R such that $T(x)^n = 0$ for all $x \in R$. Our goal will be to show that $T = 0$. Proceeding by induction trough out we assume the result to be true for any centralizer G of any prime ring B whenever $G(x)^m = 0$ for all $x \in B$, if $m < n$. We proceed assuming that $T \neq 0$. Our first result is :

LEMMA 1. For $a \in R$,$T(L(a)) \subset L(a)$ and $T(R(a)) \subset R(a)$.
PROOF: Let $x \in L(a)$ then $xa = 0$,
$0 = T(xa) = T(x)a$ for all $x \in L(a)$, therefore, $T(x) \in L(a)$ for all $x \in L(a)$. Hence $T(L(a)) \subset L(a)$.
Similarly one can show that $T(R(a)) \subset R(a)$.

LEMMA 2. If $a \in R$, then either $T(aR)a = 0$ or $L(a)T\big(L(a)\big) = 0$. Similarly, either $aT(aR) = 0$ or $T\big(R(a)\big)R(a) = 0$.
PROOF: Let $x, y \in L(a)$. Using Lemma 1 we have that $T(y)ax = 0$. Then,

$$0 = T(T(y)ax) = T(y)T(ax) \text{ for all } y \in L(a) \tag{1}$$

Since $ax \in L(a)$, then we can replace y by ax in (1) , hence, $T(ax)^2 = 0$. Now

$$0 = T(ax + y)^n = \big(T(ax) + T(y)\big)^n = T(ax)T(y)^{n-1} \text{ for all } x \in L(a) \tag{2}$$

Let $r \in R$, then by using (2) we get that , $T(arax)T(y)^{n-1} = 0$, that is, $T(ar)axT(y)^{n-1} = 0$, for all $x \in L(a)$, hence $T(ar)aL(a)T(y)^{n-1} = 0$.
If $L(a)\,T(y)^{n-1} \neq 0$, since $L(a)\,T(y)^{n-1}$ is a left ideal of a prime ring R , then $T(ar)a \in ann_l\,(L(a)T(y)^{n-1} = 0$, therefore, $T(ar)a = 0$, for all $r \in R$, hence $T(aR)a = 0$. On the other hand if $L(a)T(y)^{n-1} = 0$ for all $y \in L(a)$. Let $W = \{x \in L(a): L(a)x = 0\}$ since $T(W) \subset W$ and $T\big(L(a)\big) \subset L(a)$, T induces a centralizer on $B = L(a)/W$. By Remark 3 B is a prime ring. The fact that $L(a)T(y)^{n-1} = 0$ for all $y \in L(a)$ gives us that $T(y)^{n-1} \in W$ for all $y \in L(a)$, this gives us that $T(b)^{n-1} = 0$ for all $b \in B$, then by our induction we get that $T(b) = 0$ for all $b \in B$, this leads us to $T(L(a)) \subset W$, and hence $T\big(L(a)\big)\,L(a) = 0$.
Similarly one can show that either $T(R(a)) \subset R(a)$ or $aT(aR) = 0$.

Lemma 2 has singled out for us two classes of elements which have rather particular properties, and which prompt the following definition:

DEFINITION: $A = \{a \in R: aT(Ra) = 0\}$, and $B = \{a \in R: T(aR)a = 0\}$.
These two subsets A and B play a key rule in what is follows. Their basic algebraic behavior is expressed in the following Lemma:

LEMMA 3: A is a nonzero left ideal of R , B is a nonzero right ideal of R and $AB = 0$. Furthermore $T(A) \subset A$, $T(B) \subset B$ and $AT(A) = BT(B) = 0$.
PROOF: Since the proof for the stated properties of A and B are the same, we merely prove that $B \neq 0$ is a right ideal of R , $T(B) \subset B$ and $T(B)B = 0$.
Our first assertion is that if $a, b \in R$ are such that $L(a)T\big(L(a)\big) = 0$ and $L(b)T\big(L(b)\big) = 0$ then $L(b)T(L(a) = 0$.
To see this, let $x \in L(a)$, $z, t \in L(b)$, then ,
$0 = tT(xz) = tT(x)z$ for all $z \in L(b)$, that is $tT(x)L(b) = 0$, hence by the primeness of R we get that $tT(x) = 0$ for all $t \in L(b)$ and $x \in L(a)$. So

$$L(b)T\big(L(b)\big) = 0 \tag{3}$$

Thus our assertion has been verified .

Claim 1: $B \neq 0$ .0
Suppose that $B = 0$, then by Lemma 2 we have that $L(u)T\big(L(u)\big) = 0$ for all $u \in R$, then by (3) we have that

$L(u)T\big(L(v)\big) = 0$ for all $u, v \in R$ (4)

Pick $v \in R$ such that $L(v) \neq 0$, by Remark 2 , $T\big(L(v)\big) \neq 0$. Since $T(x)^n = 0$ for all $x \in R$ then $T(x) \in L(T(x)^{n-1})$. Let $u = T(x)^{n-1}$ in (4) then we have that $T(x)L(v) = 0$, so by Remark 1 $T\big(L(v)\big) = 0$, a contradiction since $T(L(v)) \neq 0$, hence $B \neq 0$.

Claim 2 : B is a right ideal of R .
We need to show first for $x \in R$ and $a \in B$ then $ax \in B$.
Since $T(axR)ax \subset T(aR)ax = 0$, therefore $T(axR)ax = 0$, hence $ax \in B$.
Now we shall show that $a + b \in B$ for $a, b \in B$ and $a, b \neq 0$. Since $T(bR)b = 0$, we have that $T(bRaR)b \subset T(bR)b = 0$,
$0 = T(bRaR)b = T(bR)aRb$. Since R is prime and $b \neq 0$, then $T(bR)a = 0$. Similarly one can show that $T(aR)b = 0$. Therefore,
$T\big((a+b)R\big)(a+b) = T(aR + bR)(a+b) = T(aR)a + T(aR)b + T(bR)a + T(bR)b = 0$, hence $a + b \in B$. Then B is a right ideal.

Claim 3 : $T(B) \subset B$.
Let $x \in B$, $r \in R$, then:
$T(T(x)r)T(x) = T\big(T(xr)\big)T(x) = T^2(xr)T(x) = T(T^2(xr)x)$ for all $r \in R$. hence since $x \in B$ we have that $T(T(x)R)T(x) = T(T^2(xR)x) \subset T(T(xR)x) = 0,$ then $T(T(x)R)T(x) = 0$, hence $T(x) \in B$ for all $x \in B,$ then $T(B) \subset B.$

Claim 4 : $T(B)B = 0.$
If $a, b \in B$ we saw that $T(aR)b = 0$, hence $T(abRb) = 0$, that is $T(a)bRb = 0$, since R is prime then $T(a)b = 0$ for all $a, b \in B$, hence $T(B)B = 0$.

Claim 5 : $AB = 0$
Let $a \in A$ and $b \in B$,
$0 = T(ab)^n\, T(a) = (T(a)b)^n\, T(a),$ therefore $(T(a)b)^{n+1} = 0$ for all $b \in B$, since $T(a)B$ is a right ideal , then by Remark 3 we get that $d(a)B = 0$ for all $a \in A$, thus $T(A)B = 0$, and so since A is a left ideal of R , then
$0 = T(RA)B = T(R)AB,$ and hence by Remark 1 we get that $AB = 0.$

LEMMA 4: If $t \in R$ and $t^2 = 0$, then $t \in A \cup B$.

PROOF: Suppose that $t \notin B$, by Lemma 2 , $L(t)\, T\big(L(t)\big) = 0$. However, since $t^2 = 0$, $t \in L(t)$ and $Rt \subset L(t)$, then $tT(Rt) = 0$, by definition of A this forces $t \in A$, hence $t \in A \cup B$ .Since $A \neq 0, B \neq 0$ are respectively left and right ideals of the prime ring R , $C = A \cap B \supset BA \neq 0$. ($BA \neq 0$ since if $BA = 0$, then $B \in ann_l(A) = 0$, hence $B = 0$, a contradiction). So $C \neq 0$.
Our attention will be concentrated on the nature of C .
If $a \in C$ and $t^2 = 0$ then, if $t \in A$, $ta \in AC \subset AB = 0.$ If $a \in B$ we get $at = 0$. In light of Lemma (4) we then must have that $at = 0$ or $ta = 0$. Consequently $ata = 0.$
We claim $asa = 0$ for all nilpotent elements s in R . If $s^2 = 0$ we just saw that $asa = 0$. Proceeding by induction on the index of nilpotence of s we may assume that $as^i a = 0$ for all $i > 1$. Now
$b = (1+s)a(1+s)^{-1} = (1+s)a(1 - s + s^2\, ...)$

Satisfies $b^2 = 0$, so by Lemma 4 we have $ab = 0$ or $ba = 0$, if $ab = 0$ we get that $asa = 0$; on the other hand , if $ba = 0$ we get , using $as^i a = 0$ for $i > 1$, that $asa = 0$. Hence $asa = 0$ for all nilpotent elements $s \in R$.

Now since $T(x)$ is nilpotent for every $x \in R$, then $aT(x)a = 0$. However since $a \in R \subset A$, $aT(Ra) = 0$, thus , $aRT(a) = 0$. Because R is prime we have $T(a) = 0$. Hence:

LEMMA 5: If $a \in C$, then $T(a) = 0$.

We continue with the argument we were making . Let $a \in C$, since $T(x)$ is nilpotent we have $aT(x)a = 0 = aT(x)^2a$. Because $a^2 \in C^2 \subset AB = 0$, we have that

$(aT(x) - T(x)a)^2 = aT(x)aT(x) - aT(x)^2a - T(x)a^2T(x) - T(x)aT(x)a = 0$.

But then by Lemma 4 , $aT(x) - T(x)a \in A \cup B$ for all $x \in R$. Suppose that $aT(x) - T(x)a \in A$, say; since $a \in C \subset A$, $T(x)a \in A$, hence $aT(x) \in A$. If $aT(x) - T(x)a \in B$, similarly we get $T(x)a \in B$. So, for every $x \in R$ either $aT(x) \in A$ or $T(x)a \in B$.This implies that $aT(R) \subset A$ or $T(R)a \subset B$. If $aT(R) \subset A$, then since $a \in C \subset B$, B is a right ideal ; $aT(R) \subset B$, hence $aT(R) \subset C$. Similarly , if $T(R)a \subset B$ we get $T(R)a \subset C$. So , for every $a \in C$, $aT(R) \subset C$ or $T(R)a \subset C$. This implies $CT(R) \subset C$ or $T(R)C \subset C$.

Suppose that $CT(R) \subset C$, hence $CT(R)T(A) \subset CT(A) \subset AT(A) = 0$. Now $BA \subset C$, thus $BAT(R)T(A) \subset CT(R)T(A) = 0$, because R is prime this forces $AT(R)T(A) = 0$. Consider the left ideal $AT(R)$ of R , let $x = \sum a_i T(r_i)$, $a_i \in A$, $r_i \in R$ be any element in $AT(R)$. Thus if $v = \sum a_i r_i$, then :

$T(v) = T(\sum a_i r_i) = \sum a_i T(r_i) = x$. Therefore, $0 = T(v)^n = x^n$.

In other words, every element in $AT(R)$ is nilpotent of degree at most n .By Remark 4 $AT(R) = 0$. Since $A \neq 0$ by Remark 1 we are forced to $T(R) = 0$, and so $T = 0$.

Similarly if we had supposed that $T(R)C \subset C$ we would have been led to $T(R)B = 0$ and so to $T = 0$. We have therefore proved :

THEOREM 1. If R is a prime ring and T a centralizer of R such that $T(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer , then $T = 0$.

THEOREM 2. Let R be a prime ring , $I \neq 0$ an ideal of R , and T a centralizer of R such that $T(x)^n = 0$ for all $x \in I$, where $n \geq 1$ is a fixed integer , then $T = 0$.

PROOF: Let $I \neq 0$ be an ideal of R .

Claim 1 : If R is a prime ring then I is a prime subring of R .

Since every ideal is subring then I is subring . Now, let $a, b \in I$ and $aIb = 0$, since I is ideal then $aRIb \subset aIb = 0$, by the primness of R either $a = 0$ or $b = 0$, hence I is a prime ring .

Case 1: If $T(I) \subset I$, then T induces a centralizer T of I , and since $T(x)^n = 0$ for all $x \in I$, we get by claim 1 and Theorem 1 that $T(I) = 0$, and by theorem (If $T(I) = 0$ for some one sided ideal of R , then $T(R) = 0$) , hence $T(R) = 0$.

Case 2: If $T(I) \not\subset I$, assume $T \neq 0$ on R .

Claim 2: If $T \neq 0$ a centralizer of R and $aT(x) = 0$ (or $T(x)a = 0$) for all $x \in I$, then $a = 0$. Let $r \in R$, then

$0 = a\,T(rx) = arT(x)$, for all $r \in R$, so $aRT(x) = 0$, since R is prime then either $a = 0$ or $T(x) = 0$ for all $x \in I$, if $T(x) = 0$ for all $x \in I$, then $T(I) = 0$, and so $T(R) = 0$. a contradiction . Hence $a = 0$.

Now , since $T(x)^n = 0$ for all $x \in I$, then $T(x)T(x)^{n-1} = 0$ for all $x \in I$, hence by claim 2 $T(x)^{n-1} = 0$. Continue in the same way and by using claim 2 we end up with $T(x) = 0$ for all $x \in I$, thus $T(I) = 0$, this leads to $T(R) = 0$, a contradiction . Hence $T = 0$.

Now Theorem 1 can be extended to semiprime rings:

THEOREM 3. If R is a semiprime ring and T a centralizer of R such that $T(x)^n = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $T = 0$.

PROOF: Since R is semiprime ring, $\cap P = 0$, where P is a prime ideal of R (see [9] page 115) .

Claim : $T(P) \subset P$ for every prime ideal P .

Let $a \in P$, $x \in R$;

$0 = T(ax)^n = (T(a)x)^n$ for all $x \in R$. Hence the right ideal $T(a)R$ is nil of bounded index , then R has a nilpotent ideal which it is cannot since R is semiprime , therefore , $T(a)R = 0$, hence $T(a) = 0$ for all $a \in P$, then $T(P) = 0$, so $T(P) \subset P$ for all prime ideals P of R , and so T induces a centralizer $\bar{T}$ on the prime ring $\bar{R} = R/P$, such that $\bar{T}(\bar{x})^n = 0$ for all $\bar{x} \in \bar{R}$, by Theorem 1 , $\bar{T} = 0$. Hence $\bar{T}(\bar{R}) = 0$, that is , $T(R) \subset P$ for all prime ideals P of R .

Since $\cap P = 0$, we obtain that $T(R) = 0$, hence $T = 0$.

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