On The Generalized Type and Generalized Lower Type of Entire Function in Several Complex Variables With Index Pair \((p, q)\)

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Abstract
In the present paper, we will study the generalized \((p, q)\)-type and generalized lower \((p, q)\)-type of an entire function in several complex variables with respect to the proximate order with index pair \((p, q)\) are defined and their coefficient characterizations are obtained.

Keywords: Entire function, generalized type, generalized lower type, index pair.

1 Introduction
Kumar and Gupta [1] let \(f(z_1, z_2, \ldots, z_n)\) be an entire function \(z = (z_1, z_2, \ldots, z_n) \in C^n\).

Let \(G\) be a region in \(R^n\) (positive hyper octant) and let \(G_R \subset C^n\) denote the region obtained from \(G\) by a similarity transformation about the origin, with ratio of similitude \(R\). Let \(d_t(G) = \sup_{z \in G} |z|^t\), where \(|z|^t = |z_1|^t \cdot |z_2|^t \cdot \ldots \cdot |z_n|^t\), and let \(\partial G\) denote the boundary of the region \(G\). Let

\[ f(z) = f(z_1, z_2, \ldots, z_n) = \sum_{t_1, t_2, \ldots, t_n \geq 0} a_{t_1, t_2, \ldots, t_n} z_1^{t_1} \cdot z_2^{t_2} \cdot \ldots \cdot z_n^{t_n}, \quad \|f\| = t_1 + t_2 + \ldots + t_n, \]

be the power series expansion of the function \(f(z)\). Let \(M_{f,G}(R) = \max_{z \in G_R} |f(z)|\).

To characterize the growth of \(f\), order \(\rho_G\) and type \(T_G\) of \(f\) are defined as .

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\[ \rho_G = \lim_{R \to \infty} \sup_{n \to \infty} \frac{\log \log M_{f,G}(R)}{\log R}, \text{ and } T_G = \lim_{R \to \infty} \sup_{n \to \infty} \frac{\log M_{f,G}(R)}{R^{\rho_G}}. \]

For \( R > 0 \), the maximum term \( \mu_{f,G}(R) \) of entire function \( f(z) \) is defined as (see [2] and [3])

\[ \mu_{f,G}(R) = \max \{ |a_n|d_n(G)R^\rho \}. \]

For entire function \( f(z) = \sum_{n \geq 0} a_n z^n \), A.A. Gol'dberg [4, Th.1] obtained the order and type in terms of the coefficients of its Taylor expansion as

\[ \rho_G = \lim_{R \to \infty} \sup_{n \to \infty} \frac{\|f\|_{[R,\infty]}^{1/n}}{\log |a_n|} \]

and

\[ (e\rho_G T_G)^{1/\rho_G} = \lim_{R \to \infty} \sup_{n \to \infty} \left\{ \|f\|_{[R,\infty]}^{1/n} \left[ |a_n|d_n(G)\right]^{1/n} \right\}, (0 < \rho_G < \infty) \]

where \( d_n(G) = \max_{r \in \mathbb{R}} (r^n), r' = r_1^n r_2^n \cdots r_n^n \).

The concept of \((p,q)\)-order, lower order \((p,q)\)-order, \((p,q)\)-type and lower \((p,q)\)-type of an entire function \( f(z_1, z_2, \ldots, z_n) \) having an index pair \((p,q)\), was introduced by Juneja et al. ([5], [6]). Thus \( f(z) \) is said to be of \((p,q)\)-order \( \rho_G \) and lower \((p,q)\)-order \( \lambda_G \) if

\[ \lim_{R \to \infty} \sup \frac{\log^{[p]} M_{f,G}(R)}{\log^{[q]} R} = \rho_G(p,q) \quad (1.1) \]

where \( p \) and \( q \) are integers such that \( p \geq q \geq 1 \). If \( b \leq \rho_G(p,q) \leq \infty \), where \( b = 1 \), if \( p = q \) and \( b = 0 \) if \( p > q \), then the \((p,q)\)-type \( T_G \) and lower \((p,q)\)-type \( t_G \) is given by

\[ \lim_{R \to \infty} \sup \frac{\log^{[p-1]} M_{f,G}(R)}{\log^{[q-1]} R^{\rho_G(p,q)}} = T_G(p,q) \]

and \( \log^{[m]} x = \exp^{[-m]} x = \log(\log^{[m-1]} x) = \exp(\exp^{[-m-1]} x), m = 0, \pm 1, \pm 2, \cdots \) provided that \( 0 < \log^{[m]} x < \infty \) with \( \log^{[0]} x = \exp^{[0]} x = x \).

The growth of a function \( f(z) \) can be studied in terms of its order \( \rho_G \) and type \( T_G \), but these concepts are inadequate to compare the growth of those functions which are of the same order and of infinite type. Hence, for a refinement of the above growth scale, one may utilize proximate order the concept of which is [7] as follows:

A function \( \rho_G(R) \) defined on \((0, \infty)\) is said to be a proximate order of an entire function with index pair \((p,q)\) if it satisfies the properties: \( \lim_{R \to \infty} \rho_G(R) = \rho_G \) and

\[ \lim_{R \to \infty} \rho_G(R) = 0, \text{ where } \Lambda_{[q]}(R) = \log^{[q]} R \cdots \log R R. \]

Now, we define the generalized \((p,q)\)-type \( T_G^* \) and generalized lower \((p,q)\)-type \( t_G^* \) of \( f(z) \) with respect to a given proximate order \( \rho_G(R) \) as
A proximate order \( \rho_G(R) \) is called a proximate order of an entire function \( f(z) \) with index \( (p,q) \) if \( T_G^* \) is non-zero and finite and the function \( f(z) \) is said to be of perfectly regular \( (p,q) \) growth with respect to its proximate order \( \rho_G(R) \) if \( \rho_G = T_G^* \).

In the present paper we obtain coefficient characterizations of generalized \( (p,q) \)-type \( T_G^* \) and generalized lower \( (p,q) \)-type \( t_G^* \) of the entire function \( f(z) \).

By \( \log^{(q-1)} \) \( R \) \( \rho_G(R) \) is a monotonically increasing function of \( R \) for \( 0 < R_0 < R < \infty \), so we define a single valued real function \( \chi(k) \) of \( k \) for \( k > k_0 \) such that

\[
\log \log R \rho_G(R) = \chi(k) \quad (1.3)
\]

Then we have the following:

**Lemma 1.1.** Let \( \rho_G(R) \) be a proximate order with index pair \( (p,q) \) and let \( \chi(k) \) be defined as in (1.3). Then

\[
\lim_{k \to \infty} \frac{d \log \chi(k)}{d \log k} = \frac{1}{\rho_G - A} \quad (1.4)
\]

and for every \( \eta \) with \( 0 < \eta < \infty \)

\[
\lim_{k \to \infty} \frac{\chi(\eta k)}{\chi(k)} = \eta^{\frac{1}{\rho_G - A}} \quad (1.5)
\]

where \( A = 1 \) when \( (p,q) = (2,2) \)

\[= 0 \text{ otherwise.}\]

**Proof.**

\[
\frac{d \log \chi(k)}{d \log k} = \frac{d \log \log R \rho_G(R)}{d \log \{ \log \rho_G(R) - A \} \log \log R}
\]

\[= \frac{1}{\rho_G(R) - A + \Lambda_{(q)}(R) \rho_G'(R)}.
\]

passing to the limits \( k \to \infty \) we obtain (1.4).

Again,

\[
\frac{\chi(\eta k)}{\chi(k)} = \eta^{\frac{1}{\rho_G - A}},
\]

taking limits we get (1.5).

**Lemma 1.2.** Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be an entire function having proximate order \( \rho_G(R) \) with index pair \( (p,q) \). Let \( T_G^* \) and \( t_G^* \) be the generalized \( (p,q) \)-type and generalized lower \( (p,q) \)-type of \( f(z) \) with respect to a proximate order \( \rho_G(R) \). Then

\[
\lim_{R \to \infty} \inf \frac{\log^{(p-1)} \mu f,G(R)}{\log^{(q-1)} \rho_G(R)} = \frac{T_G^*}{t_G^*} \quad (1.6)
\]

**Proof:** By the maximum term in [8] and by using the type and lower type [6], we have
For $R > 0$, the maximum term $\mu_{f,G}(R)$ of entire function $f(z)$ is defined as

$$\mu_{f,G}(R) = \mu_{f,G}(R, f) = \max \{\|f\| R^{|n|}\}$$

and

$$\lim_{R \to \infty} \sup \log \frac{\mu_{f,G}(R)}{R^{\rho_{\rho}(R)}} = T_{G}^*$$

Then from [6], we get (1.6).

2 Main Result

Theorem 2.1. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function with proximate order $\rho_{G}(R)$ and $(p, q)$-order $\rho_{G}$ with index pair $(p, q)$, then the generalized $(p, q)$-type $T_{G}^*$ of $f(z)$ with respect to the proximate order $\rho_{G}(R)$ is given by

$$T_{G}^*/M = \lim_{n \to \infty} \sup \frac{\chi(\log^{[p-2]}(\|\| a_n \|_{H}))}{\log^{[q-1]}\{-(1/\|\|) \log(\|a_{n}d_{i}(G)\|)\}}$$

where

$$M = \begin{cases} (\rho_{G} - 1)/\rho_{G} & \text{if } (p, q) = (2, 2) \\ 1/\rho_{G} & \text{if } (p, q) = (2, 1) \\ 1 & \text{if } \text{for all other index pair } (p, q) \end{cases}$$

and

$$\alpha_{\|} = \begin{cases} (t_{1}^{k_{1}}t_{2}^{k_{2}} \cdots t_{n}^{k_{n}})^{\|\|} & ; \text{for } (p, q) = (2, 1) \\ 1 & ; \text{for } 2 \leq q \leq p < \infty \\\ 0 & ; \text{at least one } t_{1}, t_{2}, \ldots, t_{n} = 0 \end{cases}$$

Proof. From (1.6) for every $\varepsilon > 0$ and for all $R > R_{0}$, $0 < R_{0} = R_{0}(\varepsilon) < R < \infty$,

$$\log M_{f,G}(R) < \exp^{[p-2]}\{T_{G}^* + \varepsilon\} \log^{[q-1]} R^{\rho_{\rho}(R)},$$

for all $R$ such that $0 < R_{0} < R < \infty$,

$$\log|a_{n}d_{i}(G)| \leq \exp^{[p-2]}\{T_{G}^* + \varepsilon\} \log^{[q-1]} R^{\rho_{\rho}(R)} - \|\| \log R.$$  \hspace{1cm} (2.2)

Now choose $R$ such that

$$\log^{[q-1]} R^{\rho_{\rho}(R) - A} = \frac{1}{T_{G}^* + \varepsilon} \log^{[p-2]}(\|\|/\rho_{G}) .$$  \hspace{1cm} (2.3)

For $(p, q) \neq (2, 2)$, (2.3) is reduced to

$$\log^{[q-1]} R^{\rho_{\rho}(R)} = \frac{1}{T_{G}^* + \varepsilon} \log^{[p-2]}(\|\|/\rho_{G}),$$

which gives that

$$k = \frac{1}{T_{G}^* + \varepsilon} \log^{[p-2]}(\|\|/\rho_{G}) \quad \text{and} \quad \log^{[q-1]} R = \chi \left( \frac{1}{T_{G}^* + \varepsilon} \log^{[p-2]}(\|\|/\rho_{G}) \right).$$
Using the results (2.2) yields
\[
\chi(\log \left[\frac{p-2}{p} \right] \| \|) \leq \frac{\chi(\log \left[\frac{p-2}{p} \right] \| \|)}{\rho_o(G(T_o^* + \varepsilon))} \left( 1 - \frac{\| \| \|/\rho_o(G(T_o^* + \varepsilon))^{p(\| \|)} \chi(\| \|/\rho_o(G(T_o^* + \varepsilon)))}{\rho_o^{\| \|/\rho_o(G(T_o^* + \varepsilon))^{p(\| \|)}} \right).
\]

Passing to limits, we have (using (1.5))
\[
\lim_{H \to \infty} \sup_{\| \|} \left[ \frac{\chi(\log \left[\frac{p-2}{p} \right] \| \|)}{\log(\log(\| \|)) - \log(G(d, G))} \right]^{\rho_o} \leq T_o^*(p \geq 3). \quad (2.4)
\]
For \((p, q) = (2, 2)\), the equation (2.3) becomes
\[
(\log R)^{\rho_o(R^{-1})} = \| \| / \rho_o(G(T_o^* + \varepsilon)).
\]
which implies that
\[
k = \| \| / \rho_o(G(T_o^* + \varepsilon)) \text{ and } \log R = \chi(\| \| / \rho_o(G(T_o^* + \varepsilon))).
\]
Hence, (2.2) is written as
\[
\frac{\chi(\| \|)}{-(1/\| \|) \log \|d_i(G)\|} \leq \frac{\chi(\| \|)}{\rho_o(G(T_o^* + \varepsilon))} \left( 1 - \frac{\| \| / \rho_o(G(T_o^* + \varepsilon))^{p(\| \|)}}{\rho_o^{\| \|/\rho_o(G(T_o^* + \varepsilon))^{p(\| \|)}} \chi(\| \|/\rho_o(G(T_o^* + \varepsilon))) \right),
\]
where
\[
p(\| \|) = 1/(\rho_o(R) - 1) \text{ and } 1 + p(\| \|) = \rho_o(R)/(\rho_o(R) - 1).
\]
Since
\[
\lim_{H \to \infty} \frac{\chi(\| \|)}{\chi(\| \| / \rho_o(G(T_o^* + \varepsilon)))} = (\rho_o T_o^*)^{\| \| (\rho_o - 1)} \quad \text{(since } \varepsilon \text{ is very small)}
\]
and
\[
\lim_{H \to \infty} \frac{\| \| / (T_o^* + \varepsilon))^{p(\| \|)}}{\rho_o^{1+p(\| \|)}} = \frac{1}{\rho_o}
\]
so
\[
\lim_{H \to \infty} \left[ \frac{\chi(\| \|)}{-\log \|d_i(G)\|} \right]^{\rho_o^{-1}} \leq \frac{\rho_o^{\rho_o}}{(\rho_o - 1)^{\rho_o^{-1}} T_o^*}. \quad (2.5)
\]
Again, for \((p, q) = (2, 1)\), (2.3) is reduced to
\[
\| \| / \rho_o(G(T_o^* + \varepsilon)) = R^{\rho_o(R)}
\]
which gives
\[
k = R^{\rho_o(R)} \iff R = \chi(k).
\]
Equation (2.2) is converted into
\[
\chi(||f||) < \frac{\chi(||f||)}{(|a_i|d_i(G))^{-1/2}} \leq e^{-1/\rho} \chi(||f||/\rho G (T_G^* + \varepsilon)).
\]

Passing to limits we have

\[
\limsup_{H \rightarrow \infty} \left( \frac{\chi(||f||)}{(|a_i|d_i(G))^{-1/2}} \right)^{\rho G} \leq T_G^* e^{\rho G}.
\]

(2.6)

Equations (2.4), (2.5) and (2.6) combine into

\[
\limsup_{H \rightarrow \infty} \left[ \frac{\chi(\log^{1/p-2}||f||/\rho H)}{\log^{1/q-2}(-1/||f|| \log |a_i|d_i(G))} \right]^{\rho G - A} \leq T_G^*/M.
\]

(2.7)

To prove the reverse inequality, let

\[
\limsup_{H \rightarrow \infty} \left[ \frac{\chi(\log^{1/p-2}||f||/\rho H)}{\log^{1/q-2}(-1/||f|| \log |a_i|d_i(G))} \right]^{\rho G - A} = \beta / M.
\]

For any \( \varepsilon > 0 \), we have for all \( ||f|| > m_0 = m_0(\varepsilon) \)

\[
|a_i|d_i(G) R^{1/2} < \exp \left[ -||f|| \exp^{1/p-2} \left( \chi \left( \frac{M}{\alpha} \log^{1/p-2}||f||/\rho H \right) \right) + ||f|| \log R \right],
\]

where \( \alpha = \beta + \varepsilon \)

So,

\[
\log \mu_{f,G}(R) < \max_{H \rightarrow 0} \left[ -||f|| \exp^{1/p-2} \left( \chi \left( \frac{M}{\alpha} \log^{1/p-2}||f||/\rho H \right) \right) + ||f|| \log R \right].
\]

(2.8)

For \( (p, q) \neq (2, 1) \) and \( (2, 2) \), using (1.4) it can be easily seen that the maximum value on the right-hand side is attained for

\[
||f|| = \exp^{1/p-2} \left( \alpha \left( \log^{1/q-2} \left( \frac{\rho G}{1 + \rho G} \log R \right) \right)^{\rho_G/(R\rho_G)} \right).
\]

Thus, for \( R \) sufficiently large we get from (2.8)

\[
\log^{1/q-1} \mu_{f,G}(R) \leq \frac{\log^{1/q-1} \left( \frac{\rho G (1 + \rho G)^{-1} \log R \right)}{(\log^{1/q-1} R)^{\rho_G/(R\rho_G)}} + o(1).
\]

Proceeding to limits

\[
T_G^* \leq \alpha.
\]

(2.9)

Consider when \( (p, q) = (2, 1) \). Let \( ||f|| = \alpha (R e^{-1/\rho})/M \), equation (2.8) is then reduced to
\[
\log \frac{\mu_{f,G}(R)}{R^\rho_G(R)} < \frac{\alpha}{\rho_G M} e^{\rho_G(R)/\rho_G}
\]
and passing to limits we get
\[
T_G^* \leq \alpha. \tag{2.10}
\]
If \((p, q) = (2,2)\), in order to get the maximum value of the right-hand side of the inequality (2.8) \(\|\|\) is given by
\[
\|\| = \alpha \left( \frac{\rho_G - 1}{\rho_G} \right)^{\rho_G(R)^{-1}} (\log R)^{\rho_G(R)^{-1}},
\]
which reduces (2.8) to
\[
\log \frac{\mu_{f,G}(R)}{(\log R)^{\rho_G(R)}} < \frac{\alpha (\rho_G - 1)^{\rho_G(R)^{-1}}}{M \rho_G^{\rho_G(R)}}.
\]
On taking limits we get
\[
T_G^* \leq \alpha. \tag{2.11}
\]
(2.9), (2.10) and (2.11) give
\[
T_G^* \leq \alpha = (\beta + \varepsilon).
\]
Since this inequality holds for every \(\varepsilon > 0\), so \(T_G^* \leq \beta\). This and (2.7) together prove the theorem.

Taking \(\rho_G(R) = \rho_F\) and \(\chi(k) = k^{(\rho_G - \varepsilon)}\), we have the following corollary which gives a formula for the \((p, q)\)-type \(T_G^*\) of the entire function \(f(z)\).

**Theorem 2.2.** Let \(f(z) = \sum_{n=0} a_n z^n\) be an entire function having the proximate order \(\rho_G(R)\) and \((p, q)\)-order \(\rho_F\) such that
\[
\phi(\|\|) = |a_n/a_{n+1}|,
\]
forms a non-decreasing function of \(\|\|\) for \(\|\| > m_0\). Then the generalized lower \((p, q)\)-type \(T_G^*\) of \(f(z)\) is given by
\[
\{a_n \}_{n=0}^\infty = \frac{\phi(\|\|)}{R^{H(\|\|)}} \text{ for } \|\| > m_0.
\]
where \(M, A, A\) and \(\|\|\) are the same as given in Theorem 2.1.

**Proof.** Since by hypothesis, \(\phi(\|\|)\) is a non-decreasing function of \(\|\|\) for \(\|\| > m_0\). We have \(\phi(\|\|) > \phi(\|\| - 1)\) for infinitely many values of \(\|\|\); otherwise \(f(z)\) ceases to be an entire function. So \(\phi(\|\|) \to \infty\) as \(\|\| \to \infty\).

When \(\phi(\|\|) > \phi(\|\| - 1)\), the term \(a_n z^n\) becomes maximum and then
\[
\mu_{f,G}(R) = \max R^{H(\|\|)}, \nu(R) = \|\| \text{ for } \phi(\|\| - 1) \leq R < \phi(\|\|).
\]
First, let \(0 < t_G^* < \infty\), in view of Lemma 1.2. , for any \(\varepsilon\) satisfying \(0 < \varepsilon < t_G^*\) and for all \(R > R_0 = R_0(\varepsilon)\) we get
\[ \log \mu_{f,G}(R) > \exp\{p-2\}[(t_G^* - \varepsilon)(\log^{q-1} R)\rho_G(R)]. \] (2.12)

Let \( a_m z^m \) and \( a_m z^m \) (\( m_1 > m_0, \phi(m_1) > R_0 \)) .

be two consecutive maximum terms of \( f(z) \). Then since \( \phi(\|f\|) \) is a non-decreasing function of \( \|f\| \) for \( \|f\| > m_0 \), we have for \( m_1 \leq \|f\| \leq m_2 - 1 \),

\[ \phi(m_1) = \phi(m_1 + 1) = \ldots = \phi(\|f\|) = \ldots = \phi(m_2 - 1) \] (2.13)

And

\[ |a_i R^{\|f\|} = |a_m| R^{m_0} \text{ for } R = \phi(\|f\|). \] (2.14)

Hence, (2.12), (2.13) and (2.14) give

\[ \log|a_i|d_i(G) + \|f\| \log \phi(\|f\|) > \exp\{p-2\}[(t_G^* - \varepsilon)(\log^{q-1} \phi(\|f\|)\rho_G(\|f\|)] \]

or,

\[ X = \frac{\{\chi(\log(\|f\|)\alpha_{\|f\|})\}^{\rho_G(A)}}{\exp[(\rho_G - A) \log(\log(\|f\|)\alpha_{\|f\|})]} \cdot \frac{\{\chi(\log(\|f\|)\alpha_{\|f\|})\}^{\rho_G(A)}}{\exp[(\rho_G - A) \log(\log(\|f\|)\alpha_{\|f\|}) \exp\{p-2\}[(t_G^* - \varepsilon)(\log^{q-1} \phi(\|f\|)\rho_G(\|f\|)] \}]} . \]

(2.15)

We note that the minimum value of the function

\[ S(R) = \frac{\{\chi(\log(\|f\|)\alpha_{\|f\|})\}^{\rho_G(A)}}{\exp[(\rho_G - A) \log(\log(\|f\|)\alpha_{\|f\|}) \exp\{p-2\}[(t_G^* - \varepsilon)(\log^{q-1} R)\rho_G(R)] \}} \]

is attained at a point \( R = R_0 \) satisfying

\[ \frac{E_{(p-2)}[(t_G^* - \varepsilon)(\log^{q-1}(R)\rho_G(R))]}{\Lambda_{(q-1)}(R)} = \|f\|/R\rho_G . \] (2.16)

For \((p, q) = (2, 1)\), (2.16) gives \( R^{\rho_G(R)} = \|f\|/(t_G^* - \varepsilon)\rho_G \iff R = \chi(\|f\|)/(t_G^* - \varepsilon)\rho_G) . \)

Hence

\[ X > \min_{0 < R < \infty} S(R) = \min_{0 < R < \infty} \frac{\{\chi(\|f\|)\alpha_{\|f\|})\}^{\rho_G}}{\exp[\rho_G(\log R - (t_G^* - \varepsilon) R^{\rho_G(R))}/\|f\|)]} \]

\[ = e\{\chi(\|f\|)/\chi(\|f\|)/(t_G^* - \varepsilon)\rho_G)\}^{\rho_G} \approx e\rho_G(t_G^* - \varepsilon) . \] (2.17)

For \((p, q) = (2, 2)\), (2.16) becomes

\[ (\log R)^{\rho_G(R)-1} = \frac{\|f\|}{\rho_G(t_G^* - \varepsilon)} \iff \log R = \chi(\|f\|)/(t_G^* - \varepsilon)\rho_G) . \]

Hence,

\[ \min_{0 < R < \infty} S(R) = \{\chi(\|f\|)/\chi(\|f\|)/(t_G^* - \varepsilon)\rho_G)\}^{\rho_G-1}\{\rho_G/(\rho_G - 1)\}^{\rho_G-1}. \]

\[ \approx \frac{\rho_G}{(\rho_G - 1)^{\rho_G-1}}(t_G^* - \varepsilon) \] (2.18)
For \((p, q) \neq (2,2)\) and \((2,1), \text{(2.16)}\) is reduced to

\[
(\log^{(q-1)} R)^{\rho_0(R)} = \frac{1}{t^*_G - \varepsilon} \log(\|R\|/\rho_G) \Leftrightarrow \log^{(q-1)} R = \chi \left( \frac{1}{t^*_G - \varepsilon} \log(\|R\|/\rho_G) \right).
\]

So

\[
\min_{\delta < R < \infty} S(R) = \left( \chi(\log(\|R\|/\rho_G))^{\rho_0} / \exp (\rho_G \log(\|R\|/\rho_G)) \right)
\]

\[
\approx \left( \frac{1}{\log(\|R\|/\rho_G)} \right)^{\rho_0}
\]

\[
\approx t^*_G - \varepsilon.
\]

(2.19)

(2.15), (2.17), (2.18) and (2.19) combine into

\[
\lim_{M \to \infty} \inf X \geq t^*_G / M.
\]

(2.20)

The inequality (2.20) is obvious if \(t^*_G = 0\). When \(t^*_G = \infty\), above arguments with an arbitrarily large number in place of \((t^*_G - \varepsilon)\) leads to

\[
\lim_{M \to \infty} \inf X = \infty.
\]

We now prove that strict inequality cannot hold in (2.20). for if it holds, then there exists a number \(\delta(\delta > t^*_G)\) such that

\[
\delta = \lim_{M \to \infty} \inf \left[ \frac{\chi(\log(\|R\|/\rho_G))}{\log(\|R\|/\rho_G)} \log|a_i d_i(G)| \right]^{\rho_0 - A}.
\]

Let \(\delta_i\) be such that \(\delta > \delta_i > t^*_G\), then for all \(\|R\| > m_0\)

\[
\log|a_i d_i(G)| > -\|R\| \exp^{(q-2)} \left[ \frac{\chi(\log(\|R\|/\rho_G))}{(\delta_i / M)^{1/(\rho_0 - A)}} \right]^{\rho_0 - A}.
\]

Therefore, for sufficiently large \(R\) and \(\|R\|\) we have

\[
\log M_{f,G}(R) > -\|R\| \exp^{(q-2)} \left[ \frac{\chi(\log(\|R\|/\rho_G))}{(\delta_i / M)^{1/(\rho_0 - A)}} \right] + \|R\| \log R.
\]

(2.21)

For \((p, q) = (2,1)\), choose \(\|R\| = [\rho_0 \delta_i R^{\rho_0(R)}]\), then in view of Lemma 1.1,

\[
\log M_{f,G}(R) > -\|R\| \log \left[ \frac{\chi(\|R\|/\rho_G)}{(e \rho_0 \delta_i)^{1/(\rho_0)}} \right] + \|R\| \log \chi(\|R\|/\rho_0 \delta_i)
\]

or,
\[
\log \frac{M_{f,G}(R)}{R^{\rho_G(R)}} > \delta_1.
\]

Passing to limits
\[
t^*_G \geq \delta_1.
\] (2.22)

In case \((p, q) = (2, 2)\), choose
\[
(\log R)^{\rho_G(R)-1} = \frac{M_1}{\delta_1((\rho_G - 1) / \rho_G)^{\rho_G(R)-1}} = k,
\]

Then (2.21) is reduced to
\[
\log M_{f,G}(R) > \left\| \log R - \chi(\|/\|)(M/\delta_1)^{\rho_G(R)-1} \right\|
\]
\[
\approx \frac{\|/\|}{\rho_G} \log R
\]

or,
\[
\frac{\log M_{f,G}(R)}{(\log R)^{\rho_G(R)}} > \delta_1
\]

which gives on passing to limits
\[
t^*_G \geq \delta_1.
\] (2.23)

Further, consider \((p, q) \neq (2, 1)\) and (2, 2) if \(\|/\|\) is given by
\[
\log^{[p-2]}(\|/\|/\rho_G) = \delta_1(\log^{[q-1]}R/e^e)^{\rho_G(R)/e^e} \Leftrightarrow \log^{[q-1]}R/e^e = \chi\left(\frac{\log^{[p-2]}(\|/\|/\rho_G)}{\delta_1}\right);
\]
then
\[
\log M_{f,G}(G) > \left\| \log R - \exp^{[q-2]}\left[ \frac{\chi(\log^{[p-2]}(\|/\|))}{\delta_1^{[p-2]}} \right] \right\|
\]
\[
= \left\| \varepsilon + \exp^{[q-2]}\left[ \frac{\chi(\log^{[p-2]}(\|/\|/\rho_G))}{\delta_1^{[p-2]}} \right] - \exp^{[q-2]}\left[ \frac{\chi(\log^{[p-2]}(\|/\|))}{\delta_1^{[p-2]}} \right] \right\|
\]

or,
\[
\frac{\log^{[p-1]} M_{f,G}(R)}{(\log^{[q-1]} R)^{\rho_G(R)}} > \delta_1 + o(1).
\]

Proceeding to limits we have
\[
t^*_G \geq \delta_1.
\] (2.24)

So (2.22), (2.23) and (2.24) are formed in to
\[
t^*_G \geq \delta_1.
\]

which is a contradiction. Hence the proof of the theorem is complete.
Corollary 2.3. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function having the \( (p, q) \)-order \( \rho_G \) and lower \( (p, q) \)-type \( t_G(0 \leq t_G < \infty) \) such that \( \phi(\|r\|) \) is non-decreasing function of \( \|r\| \) for \( \|r\| > m_0 \), then

\[
t_G/M = \lim_{\|r\| \to \infty} \inf \log^{(p-2)\|r\|\alpha} \|r\|^{1/(q-2)(-1/\|r\|\log|a_n|)}^{\rho_G-A}.
\]

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References