R- Annihilator – Small Submodules

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Abstract
Let R be an associative ring with identity and let M be a unital left R-module. We call a submodule K of M, R-annihilator –small if \( K+T=M \), where T is a submodule of M, implies that \( \text{ann}(T)=0 \), where \( \text{ann}(T) \) indicates annihilator of T in R. The sum \( A_M \) of all such submodules of M contains the Jacobson radical J(M) and the singular submodule Z(M). When M is finitely generated and faithful, we study \( A_M \) and \( K_M \) in this paper. Conditions when \( A_M \) is R-annihilator-small and \( K_M=A_M \), \( J(M) \subseteq A_M \) and \( Z(M) \subseteq A_M \) are given.

Keywords: Small submodules, annihilators, R-annihilator- small submodules.

1-Introduction
Throughout this paper all rings are associative ring with identity and modules are unital left modules.

In [1], Nicholson and Zhou defined annihilator –small right(left) ideals as follows: a left ideal A of a ring R is called annihilator-small if \( A+T=R \), where T is a left ideal, implies that \( r(T)=0 \), where \( r(T) \) indicates the right annihilator.

Kalati and Keskin consider this problem for modules in [2] as follows:- let M be an R-module and \( S=\text{End}(M) \). A submodule K of M is called annihilator-small if \( K+T=M \) , T a submodule of M, implies that \( r(T)=0 \), where \( r(T) \) indicates the right annihilator of T over \( S=\text{End}(M) \), where \( r(T) = \{ f \in S : f(T) = 0, \forall t \in T \} \).

These observation lead us to introduce the following concept . A submodule N of an R-module M is called R-annihilator- small if \( N+T=M \), T a submodule of M, implies that \( \text{ann}_R(T) = 0 \), where \( \text{ann}_R(T) = \{ r \in R : r \cdot T = 0 \} \).

In fact, the set \( K_M \) of all elements k such that \( R_k \) is semisubmodule and annihilator-small .And contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful.

The submodule \( A_M \) generated by \( K_M \) is a submodule of M analogue of the Jacobson radical that contains every R-annihilator –small submodules . In this work we give some basic properties of R-annihilator-small submodules and various.

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Characterizations
We abbreviate the Jacobson radical as \(\text{Rad}(M)\) and the singular submodule as \(\text{Z}(M)\) for any \(R\)-module \(M\). The notations \(N \leq^e M\) and \(N \leq^{\max} M\) mean respectively that a submodule \(N\) of \(M\) is essential and maximal in the module \(M\). See [1] / [2].

2. \(R\)- annihilator – small submodules.
In this section we give various characterizations of \(R\)- annihilator-small submodules. We start this section by the following definition:

**Definition (2.1)** We say that a submodule \(N\) of a module \(M\) is \(R\)- annihilator – small in \(M\) if \(N + X = M\), \(X\) a submodule of \(M\), implies that \(\text{ann}_X = 0\), we write \(N \ll^a M\) in this case.

**Examples (2.2):**
1- \(R\)-a small submodule need not be small, for example, consider \(Z\) as \(Z\)- module, every proper submodule of \(Z\) is \(Z\)-small. But \(\{0\}\) is the only small submodule of \(Z\).
2- A small submodule need not be \(R\)-a small, for example, consider \(Z_4\) as \(Z\)- module, \(Z_4\) has no \(Z\)-a small submodule. But \(\{0, 2\}\) are small in \(Z_4\).

Now, we give some basic properties of \(R\)-a small submodules

**Proposition (2.3):** Let \(A\) and \(B\) be submodules of a module \(M\) such that \(A \leq B\). If \(A \ll^a B\), then \(A \ll^a M\).

**Proof:** Let \(M = A + X\), \(X \leq M\). By Modular Law, \(B = A + (X \cap B)\).
Since \(A \ll^a B\), then \(\text{ann}(X \cap B) = 0\). One can easily show that \(\text{ann} X = \text{ann}(X \cap B)\) and therefore \(\text{ann} X = 0\). Thus \(A \ll^a M\).

**Proposition (2.4):** Let \(A\) and \(B\) be sub modules of a module \(M\) such that \(A \leq B\). If \(B \ll^a M\), then \(A \ll^a M\).

**Proof:** Let \(M = A + X\). Then \(M = B + X\). Since \(B \ll^a M\), then \(\text{ann} X = 0\).

**Proposition (2.5):** Let \(M\) and \(N\) be two \(R\)- modules and \(f : M \rightarrow N\) be an epimorphism. If \(H \ll^a N\), then \(f^{-1}(H) \ll^a M\).

**Proof:** Let \(M = f^{-1}(H) + X\), \(X \leq M\). Then \(f(M) = f(f^{-1}(H) + X) = f(f^{-1}(H)) + f(X)\). Since \(f\) is an epimorphism, then \(N = H + f(X)\). But \(H \ll^a N\), therefore \(\text{ann} f(X) = 0\). Clearly \(\text{ann} X \subseteq \text{ann} f(X)\) and hence \(\text{ann} X = 0\). Thus \(f^{-1}(H) \ll^a M\).

**Note.** Let \(f : M \rightarrow N\) be an epimorphism. Then the image of \(R\)-a small submodule of \(M\) need not be \(R\)-a small in \(N\) as the following example shows:

Consider \(Z\) and \(Z_4\) as \(Z\)- modules and let \(\pi : Z \rightarrow Z_4\) be the natural epimorphism. \(\{0\} \ll Z\). But \(\pi(\{0\}) = \{0\}\) is not \(Z\)-a small in \(Z_4\), where \(Z_4 = \{0\} + Z_4\) and \(\text{ann} Z_4 = 4Z \neq 0\).

**Note.** The sum of two \(R\)-a small submodules of a module \(M\) need not be \(R\)-a small submodule.

For example, In \(Z\) as \(Z\)- module, each of \(2Z\) and \(3Z\) are \(Z\)-a small submodule of \(Z\). But \(Z = 3Z + 2Z\) is not \(Z\)-a small in \(Z\).
We prove the following:

**Proposition (2.6):** Let \(M_1\) and \(M_2\) be a \(R\)- modules. If \(K_1 \ll^a M_1\) and \(K_2 \ll^a M_2\), then \(K_1 \otimes K_2 \ll^a M_1 \otimes M_2\).

**Proof:** Let \(P_i : M_i \oplus M_2 \rightarrow M_i\), \(i = 1, 2\) be the projection maps.
Since \(K_1 \ll^a M_1\) and \(K_2 \ll^a M_2\), then by prop(2.5)
\(K_1 \otimes K_2 = P_1^{-1}(K_1) \ll^a M_1 \otimes M_2\) and \(M_1 \otimes K_2 = P_2^{-1}(K_2) \ll^a M_2 \otimes M_2\).
By prop(2.4)
\((K_1 \otimes K_2) \cap (M_1 \otimes K_2) = K_1 \otimes K_2 \ll^a M_1 \otimes M_2\).

**Remark (2.7):** Let \(R\) be an integral domain and let \(M\) be a torsion free module. Then every proper submodule of \(M\) is \(R\)-a small in \(M\).

**Proof:** Clear

**Remark (2.8):** Let \(M\) be a faithful \(R\)-module, then every small submodule is a small.

**Proof:** Clear

**Proposition (2.9):** Let \(M\) be a faithful \(R\)- module and \(N\) be a submodule of \(M\) such that \(\text{ann} N \leq^e R\), then \(N \ll^a M\).

**Proof:** Let \(M = N + K\), Then \(0 = \text{ann} M = \text{ann}(N + K)\). One can easily show that \(\text{ann}(N + K) = \text{ann} N \cap \text{ann} K\). Then \(0 = \text{ann} N \cap \text{ann} K\). But \(\text{ann} N \leq^e R\), therefore \(\text{ann} K = 0\).
Thus \(N \ll^a M\).

**Proposition (2.10):** Let \(R\) be an integral domain and \(M\) be a faithful \(R\)-module. Then every submodule \(N\) of \(M\) such that \(\text{ann} N \neq 0\) is \(R\)-a small.

**Proof:** Let \(M = N + K\), Then \(0 = \text{ann} M = \text{ann}(N + K) = \text{ann} N \cap \text{ann} K\).
Since \( \text{ann} N \neq 0 \) and \( R \) is integral domain, then \( \text{ann} N \leq^e R \) by [4]. Therefore \( \text{ann} K = 0 \). Thus \( N \) is \( R \)-a-small submodule of \( M \).

**Proposition (2-11)**: Let \( R \) be an integral domain and \( M \) be a faithful and torsion module.

Then every finitely generated submodule \( N \) of \( M \) is \( R \)-a-small.

**Proof**: Let \( N = R x_1 + R x_2 + \ldots + R x_n \) be a submodule of \( M \) and \( M = N + K \).

Then \( 0 = \text{ann} M = \text{ann}(N + K) = \text{ann} N \cap \text{ann} K \) since \( M \) is torsion, \( \forall i = 1, 2, \ldots, n \).

But \( R \) is an integral domain, therefore \( \text{ann} R x_i \) is essential in \( R, \forall i = 1, 2, \ldots, n \) and hence \( \cap_{i=1}^{n} \text{ann} R x_i \) is essential in \( R, \) by [3].

Hence \( \text{ann} K = 0 \). Thus \( N \leq ^a M \).

**3-Characterizations of \( R \)-annihilator-small submodules**

In this section we give various characterizations of \( R \)-annihilator-small submodules.

Compare the following with prop 2 in [1].

**Proposition (3-1)**: Let \( M \) be a finitely generated module and \( K \leq ^a M \), then \( K + \text{Rad} M + Z(M) \leq ^a M \).

**Proof**: Let \( M = R m_1 + R m_2 + \ldots + R m_n, m_i \in M, \forall i = 1, 2, \ldots, n \) and \( M = K + \text{Rad} M + Z(M) + X \). Since \( M \) is finitely generated, then \( \text{Rad} M \leq M \), by [3].

And hence \( M = K + Z(M) + X \). So \( m_i = k_i + z_i, k_i \in K, z_i \in Z(M), \forall i = 1, 2, \ldots, n \).

It’s easy to show that \( M = K + R z_1 + R z_2 + \ldots + R z_n + X \). But \( k \leq ^a M \), therefore \( \text{ann} (R z_1 + R z_2 + \ldots + R z_n + X) = 0 \).

Hence \( (\cap_{i=1}^{n} \text{ann}(R z_i)) \cap \text{ann} X = 0 \). Since \( z_i \in Z(M) \), \( \forall i = 1, 2, \ldots, n \).

Then \( \text{ann} (z_i) \leq^e R, \forall i = 1, 2, \ldots, n \) and hence \( \cap_{i=1}^{n} \text{ann}(z_i) \leq^e R \), by [4].

So \( \text{ann} X = 0 \). Thus \( K + \text{Rad} M + Z(M) \leq ^a M \).

By the same way one can prove the following Proposition.

**Proposition (3-2)**: Let \( M \) be a module and \( K \leq ^a M \). If \( \text{Rad} M \leq M \) and \( Z(M) \) is finitely generated, then \( K + \text{Rad} M + Z(M) \leq ^a M \).

The following theorem gives a characterizations of cyclic \( R \)-a-small submodules.

**Theorem (3.3)**: Let \( M = \sum_{a \in A} R x_a \) be a module and \( k \in M \). Then the following statements are equivalent:

1. \( \text{Rk} \leq ^a M \)
2. \( \cap_{a \in A} \text{ann}(x_a - r_a k) = 0, \forall r_a \in R \)

**Proof**: (1) \( \Rightarrow \) (2)

Let \( r_a \in R \). For each \( a \in A \). Then \( x_a = x_a - r_a k + r_a k, \forall a \in A \).

Then \( M = \sum_{a \in A} R (x_a - r_a k) + \text{Rk} \). Since \( \text{Rk} \leq ^a M \), then \( 0 = \text{ann}(\sum_{a \in A} R (x_a - r_a k)) = \cap_{a \in A} \text{ann}(x_a - r_a k) \).

(2) \( \Rightarrow \) (1)

Let \( M = \text{Rk} + B \). Then for each \( a \in A \), \( x_a = r_a k + b_a, r_a \in R \) and \( b_a \in B \).

Now let \( t \in \text{ann} B \). Then \( x_a = t r_a k + t b_a \). Since \( t b_a = 0 \), then \( t (x_a - r_a k) = 0, \forall a \in A \).

So \( t \in \text{ann} (x_a - r_a k) = 0, \forall a \in A \).

Hence \( t \in \cap_{a \in A} \text{ann}(x_a - r_a k) = 0 \).

Now, compare it with lemma 4 in [1].

**Theorem (3.4)**: Let \( R \) be a commutative ring, \( M = \sum_{a \in A} R x_a \) be a module and \( k \in M \). Then the following statements are equivalent:

1. \( \text{Rk} \leq ^a M \)
2. \( \cap_{a \in A} \text{ann}(x_a - r_a k) = 0, \forall r_a \in R \)
3. There exists \( a \in A \) s.t \( b x_a \in \text{Rbk} \), \( \forall 0 \neq b \in R \)

**Proof**: (1) \( \Rightarrow \) (2) By Th (3.3)

(2) \( \Rightarrow \) (3) Let \( b = 0 \neq b \in R \). Assume that \( b x_a \in \text{Rbk}, \forall a \in A \).

Then \( b x_a = r_a k b = b r_a k, r_a \in R \). Therefore \( b \in \text{ann}(x_a - r_a k), \forall a \in A \).

And hence \( 0 \neq b \in \cap_{a \in A} \text{ann}(x_a - r_a k) = 0 \) which is a contradiction.

(3) \( \Rightarrow \) (2) Let \( b \in \cap_{a \in A} \text{ann}(x_a - r_a k) \) and hence \( b \in \text{ann}(x_a - r_a k), \forall a \in A \).
Let $R$ be a commutative ring, let $M=\sum_{\alpha \in \Lambda} R x_\alpha$ be a module and $K \subseteq M$. Then the following statements are equivalent:

1. $K \ll a M$
2. $\bigcap_{\alpha \in \Lambda} \text{ann}(x_\alpha - k_\alpha) = 0$, $\forall k_\alpha \in K$

**Proof:**

1) $\Rightarrow$ 2) Let $k_\alpha \in K$, $\forall \alpha \in \Lambda$. Then $x_\alpha = x_\alpha - k_\alpha + k_\alpha$, $\forall \alpha \in \Lambda$

And hence $M = \sum_{\alpha \in \Lambda} R(x_\alpha - k_\alpha) + K$. But $K \ll a M$,

therefore $0 = \text{ann}(\sum_{\alpha \in \Lambda} R(x_\alpha - k_\alpha)) = \bigcap_{\alpha \in \Lambda} \text{ann}(x_\alpha - k_\alpha)$.

2) $\Rightarrow$ 1) Let $M = K + \Lambda$. Then for each $\alpha \in \Lambda$, $x_\alpha = k_\alpha + a_\alpha$, $a_\alpha \in A$, $k_\alpha \in K$.

Hence $a_\alpha = x_\alpha - k_\alpha$, for each $\alpha \in \Lambda$. So $M = \sum_{\alpha \in \Lambda} R(x_\alpha - k_\alpha) + K$.

Now let $t \in \text{ann} \Lambda$. Therefore $t(x_\alpha - k_\alpha) = 0$, $\forall \alpha \in \Lambda$.

So $t \in \bigcap_{\alpha \in \Lambda} \text{ann}(x_\alpha - k_\alpha) = 0$. Thus $\text{ann} \Lambda = 0$ and $K \ll a M$.

**Definition (3.6):** Let $M$ be a module and $k \in M$, we say that $k$ is R-a-small in $M$ if $Rk \ll a M$.

Let $K_M = \{ k \in M : Rk \ll a M \}$.

Note that $Z(M) \subseteq K_M$ and $\text{Rad}(M) \subseteq K_M$, when $M$ is finitely generated and faithful, by prop (3.1) and prop (2.4).

Note. $K_M$ is not closed under addition in general. For example, consider $Z$ as Z-module. The sum of R-a-small need not be R-a-small. Clearly that $3Z \ll a Z$ and $2Z \ll a Z$. But $Z = 3Z + 2Z$ is not R-a-small in $Z$.

**Remark (3.7):** Let $M$ be a module and $k \in K_M$. Then $Rk \subseteq K_M$.

**Proof:** Let $r \in R$. Clearly that $Rr \subseteq Rk \ll a M$. By prop (2.4), $Rr \ll a M$ and hence $Rr \in K_M$.

Thus $Rk \subseteq K_M$.

**Remark (3.8):** Let $M$ be a module and $A \ll a M$, then $A \subseteq K_M$.

**Proof:** Let $x \in A$, then $Rx \subseteq K \ll a M$ and hence $Rx \ll a M$, by prop (2.4). Thus $x \in K_M$.

As we have seen, the sum of R-a-small submodules need not be R-a-small (consider $3Z + 2Z$ in $Z$) With this in mind we define.

**Definition (3.9):** Let $M$ be a module. Let R-a-small submodule $A_M$ of $M$ be the sum of R-a-small submodules of $M$. If $M$ has no R-a-small submodule, we write $A_M = M$.

It is clear that $K_M \subseteq A_M$ in every module, but this may not be equality (consider $Z$ as Z-module).

**Proposition (3.10):** Let $M$ be a module such that $K_M \neq \emptyset$. Then

1. $A_M$ is a submodule of $M$, $A_M$ contains every R-a-small submodule of $M$.
2. $A_M = \{ k_1 + k_2 + \ldots + k_n | k_i \in K_M \}$ for each $i$, $n \geq 1$.
3. $A_M$ is generated by $K_M$.
4. If $M$ is finitely generated, then $\text{Rad}(M) \subseteq A_M$ and $Z(M) \subseteq A_M$.

**Proof:**

**Proposition (3.11):** Let $M$ be a module such that $K_M \neq \emptyset$. Then the following statements are equivalent:

1. $K_M$ is closed under addition, that is a finite sum of R-a-small elements is R-a-small.

**Proof:**

1) $\Rightarrow$ 2) Let $k_1 + k_2 + \ldots + k_n \in A_M$, $k_i \in K_1$ and $K_i \ll a M$, $\forall i = 1, \ldots, n$.

Then $Rk_1 \ll a M$, by prop (2.4). Hence $k_i \in K_M$, $\forall i = 1, \ldots, n$.

By our assumption, $k_1 + k_2 + \ldots + k_n \in K_M$. Thus $A_M = K_M$.

2) $\Rightarrow$ 1) Assume that $A_M = K_M$ and let $x, y \in K_M$. Since $K_M \subseteq A_M$, then $x, y \in A_M$.

But $A_M$ is a submodule of $M$, by prop (3.10), therefore $x + y \in A_M = K_M$. Thus $K_M$ is closed under addition.

**Proposition (3.12):** Let $M$ be a module such that $K_M \neq \emptyset$. Consider the following statements:

1. $A_M \ll a M$.
2. If $K \ll a M$ and $L \ll a M$ then $K + L \ll a M$.
3. $K_M$ is closed under addition, that is the sum of R-a-small elements is R-a-small.

Then (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). If $M$ is finitely generated, then (1) $\iff$ (2)
Proof:- (1) $\rightarrow$ (2) Assume that $A_M \ll^a M$ and let $K \text{ and } L$ be $R$-a–small submodules of $M$.
Then $K+L \subseteq A_M$. But $A_M \ll^a M$, therefore $K+L \ll^a M$, by prop (2.4).
(1) $\rightarrow$ (3) Clear
(3) $\rightarrow$ (4) By prop (3.11)
To show that $K+L$.
(2)

$\text{Claim that } Ra \ll^a M \text{. Assume not. Then } M=Ra+X \text{, } X \subseteq M \text{ and } \text{ann}X \neq 0$.

Since $A_M \ll^a M$, then $M \neq A_M + X$. But $M$ is finitely generated, then there exist a maximal submodule such that $A_M + X \subseteq B$, by [3].

Now, if $a \in B$ we get $B=M$ which is a contradiction. So $a \notin B$.
But $a \in \cap \{ W \mid W \text{ maximal submodule of } M \text{ with } A_M \subseteq W \}$, which is a contradiction.
Thus $Ra \ll^a M$ and hence $a \in A_M$.

Reference