Centralizers on Prime and Semiprime Γ-rings

Sameer Kadem
Department of Computer Techniques Engineering, Dijlah University College, Baghdad, Iraq.

Abstract
In this paper, we will generalize some results related to centralizer concept on prime and semiprime Γ-rings of characteristic different from 2. These results relating to some results concerning left centralizer on Γ-rings.

Keywords: Semiprime Γ-ring, Centralizers.

تمركزات على الحلقات الأولية وشبه أولية من النمط كاما

سمير كاظم
قسم هندسة تقنيات الحاسوب، كلية دجلة الجامعة، بغداد، العراق.

الخلاصة
في هذه البحث، سوف نعمم بعض النتائج المتعلقة بمفهوم التمركز على الحلقات الأولية وشبه الأولية من النمط كاما التي تمثلها لا يساوي 2. هذه النتائج متعلقة مع بعض النتائج للتمركز الأيسر على الحلقات من النمط كاما.

1. Introduction
Nobusawa in [1] presented the idea of a Γ-ring, the concept of Γ-ring is more general of the Ring Barnes in [2] the definition of the Γ-ring with less conditions. On the basis of these two definitions many researchers in pure mathematics have made working on Γ-ring sense Barnes and Nobusawa see [3-6], which parallel results in the Ring theory. Barnes in [2] defined it as following: suppose N and Γ be an additive abelian groups, if there exists a map from N×Γ×N to N, for all a, b, c ∈ N and γ, δ ∈ Γ satisfying the following conditions:

1. ayb ∈ N.
2. (a+b)γc=ayc+bγc, a(γ+δ)b=ayb+aδb and ay(b+c)=ayb+ayc
3. (ayb)δc=ay(bδc).

Then N is called Γ-ring.

Some preliminaries of Γ-rings was given by S. Kyuno [7] as following: "Let I be a non-zero subset of a Γ-ring N, then I is called a left (right) ideal, if I be an additive subgroup of N and NGI (IGN), if I be a left and right ideal then I is called an ideal of N. N is called 2-torsion free if 2a=0 obtain a=0, a∈N. A Γ-ring N is said to be prime if aΓN=(0) with a, b ∈ N, obtain a=0 or b=0 and it semiprime if aΓN=(0) with a∈N, obtain a=0. A Γ-ring N is called commutative if ayb=bya, for all a, b ∈ Γ and γ∈Γ. The subset Z(N)={a∈N| ayb=bya, for all a∈N and γ∈Γ } of a Γ-ring N is called center of N." An additive mapping T:N→N is called left (right) centralizer if T(ayb)=T(a)yb (T(ayb)=ayT(b)) for all a, b ∈ N and γ∈Γ, and T is called Jordan left (right) centralizer if T(ayb)=T(a)ya (T(ayb)=ayT(a)) for all a ∈ N and γ∈Γ. If T are both left and right centralizer then T is called centralizer. Also the element (ayb-bya) ∈ N is called the commutator.

Email: sameer.kadem@duc.edu.iq
of $a$ and $b$ with respect to $\gamma$ which is denoted by $[a,b]_\gamma$. In [8] S. Chakraborty and A.C. Paul show that if $N$ is a $\Gamma$-ring for all $a,b,c \in N$ and $\gamma,\delta \in \Gamma$, then

i. $[a+b,c]_\gamma=[a,c]_\gamma+[b,c]_\gamma$

ii. $[a,b+c]_\gamma=[a,b]_\gamma+[a,c]_\gamma$

iii. $[a_\delta b,c]_\gamma=a_\delta[b,c]_\gamma+[a,c]_\gamma\delta b+a_\delta c_\gamma b-a_\gamma c_\delta b$

In this paper we assume that $a_\delta b=a_\gamma c_b$ which represent by $(\ast)$ then from equation (iii), we get $[a_\delta b,c]_\gamma=a_\delta[b,c]_\gamma+[a,c]_\gamma\delta b$. In [9] M.F. Hoque and A.C. Paul proved that if $N$ be a semiprime $\Gamma$-ring of characteristic different from 2 with condition $(\ast)$ then the Jordan left centralizer is left centralizer on $N$ and they proved if $N$ be a $\Gamma$-ring of characteristic different from 2 with condition $(\ast)$ then the Jordan centralizer is a centralizer on $N$. In this paper we show that if $N$ be a 2-torsion free semi-prime $\Gamma$-ring with condition $(\ast)$, $I$ be an ideal of $N$ and $T:N \rightarrow N$ be a Jordan left centralizer on $I$, then $N$ contains a central ideal ideal and if is a prime $\Gamma$-ring of characteristic different from 2 with the same above hypotheses then $N$ is commutative $\Gamma$-ring.

2. The Results

To prove the main result, we begin with some lemmas:

**Lemma 2.1.** [9] Suppose $N$ be a semi-prime $\Gamma$-ring, if $a,b,c \in N$ and $\gamma,\delta \in \Gamma$, such that $a_\gamma c_\delta b=0$ for all $c \in N$, then $a_\gamma b=\gamma a_\delta b=0$.

**Lemma 2.2.** [9] Suppose $N$ be a semi-prime $\Gamma$-ring and $F:N \times N \rightarrow N$, a bi-additive mapping. If $F(a,b)\gamma c_\delta \delta f(a,b)=0$ for all $a,b,c \in N$ and $\gamma,\delta \in \Gamma$, then $F(a,b)\gamma c_\delta \delta f(u,v)=0$ for all $a,b,c,u,v \in N$.

**Lemma 2.3.** Suppose $N$ be a semi-prime $\Gamma$-ring with condition $(\ast)$ and $x$ be a fixed element in $N$. If $x_\delta[a,b]=0$, for all $a,b \in N$ and $\delta,\gamma \in \Gamma$, then $N$ have central ideal $I$, such that $x \in I \subset N$.

**Theorem 2.4.** Suppose $N$ be a 2-torsion free semi-prime $\Gamma$-ring with condition $(\ast)$, $I$ be an ideal of $N$ and $T:N \rightarrow N$ be a Jordan left centralizer on $I$, then $N$ contains a central ideal.

**Proof:**

for all $a \in I$ and $\gamma \in \Gamma$, then $T(a\gamma a)=T(a)\gamma a$ (1)

if we replace $a$ by $(a+b)$ in (1), we get for all $\gamma \in \Gamma$ $T(a\gamma b+b\gamma a)=T(a)\gamma b+T(b)\gamma a$ (2)

In (2) replace $b$ by $a\gamma b+b\gamma a$ and $\gamma$ by $\delta$, for all $a \in I$ and $\gamma,\delta \in \Gamma$, we obtain

$T(a)\gamma b+T(b)\gamma a=a_\delta[a\gamma b+b\gamma a]+T(a)\gamma b+T(b)\gamma a$ (3)

Calculate (3) By deferent way then

$T(a)\gamma b+T(b)\gamma a= T(a)\gamma b+T(b)\gamma a+2T(a)\gamma b+T(b)\gamma a$ (4)

By subtracting Eq. 3 from Eq. 4 resulting in

$T(a)\gamma b+T(b)\gamma a$ (5)

In Eq. 5 replace $a$ by $a+c$ for all $c \in I$, we obtain

$T(a)\delta b(c)+T(c)\delta b(a+c)=T(a)\delta b(c)+T(c)\delta b(a+c)$ (i)

And we can show that

$T(a)\delta b(a+c)=T(a)\delta b(a)+T(c)\delta b(c)+T(c)\delta b(c)$ (ii)

From (i) and (ii), we get

$T(a)\delta b(c)+T(c)\delta b(c)$ (6)

Suppose that $J=T(\gamma a_\delta b+c_\delta a_\gamma b)$ for all $a,b,c \in I$ and $\gamma,\delta,\alpha,\beta \in \Gamma$, and calculate $J$ by two deferent way as follows:

By using Eq. 5 resulting in

$J=T(a)\gamma b+c_\delta a_\gamma b+T(b)\gamma a_\delta c_\delta a$ (7)

And by Eq. 6 resulting in

$J=T(a)\gamma b+c_\delta a_\gamma b+T(b)\gamma a_\delta c_\delta a$ (8)

By subtracting Eq. 8 from Eq. 7 resulting in

$0=T(a)\gamma b-c_\delta a_\gamma b+T(b)\gamma a_\delta c_\delta a$ (9)

Suppose the following bi-additive map $F(a,b)=T(a)\gamma b-T(a)\gamma b$, and we can show that $F(a,b)=F(b,a)$. So Eq. 9 become $0=F(a,b)\delta c_\delta a_\gamma b+F(b,a)\delta c_\delta a_\gamma b$ and
F(a, b) δcα[a, b]γ = 0, using Lemma 2.2. we have F(a, b) δcα[u, v]γ = 0 in this equation fix some a, b ∈ I and let F= F(a, b), then Fδcα[u, v]=0, for all u, v ∈ I that mean by lemma 2.1. Fδ [u, v]=0 and by lemma 2.3. we get N have central ideal.

From Theorem 2.4. and using some lemmas in Γ-rings corresponding to lemmas in the Rings Theory we can prove some results.

**Lemma 2.5.** Suppose N be a semi-prime Γ-ring with condition (*) and I be a left ideal of N then \( \Gamma(I) \subseteq Z(N) \).

**Proof:** if \( a \in Z(I) \), since I is left ideal then \( xγa \subseteq I \), for all \( x \in N \) and \( γ \in Γ \) also \( 0 = [a, xγa] \), that lead to \( 0 = [a, x]_γa \) \( \gamma \in Γ \)

By Eq. 1 for all \( y \in N \), then
\[
0 = [a, x]_γa y \quad \text{for all } \gamma \in Γ
\] (2)

In Eq. 1 replace \( x \) by \( xγy \) we obtain
\[
0 = [a, x]_γa γy \quad \text{for all } \gamma \in Γ
\] (3)

From Eq. 2 and Eq. 3 we obtain
\[
0 = [a, x]_γa γy = 0 \quad \text{for all } \gamma \in Γ
\] (4)

In Eq. 4 replace \( y \) by \( xγa \), for all \( \alpha \in Γ \), we get
\[
0 = [a, x]_γa γx = 0 \quad \text{for all } x, y \in N \text{ and } \alpha \in Γ
\]

**Lemma 2.6.** Suppose N be a semi-prime Γ-ring with condition (*) and let I be a non-zero left ideal of N. if I be a commutative as a Γ-ring, then \( Z(N) \), if in addition N is a prime Γ-ring, then N must be commutative.

**Proof:**

By Lemma 2.5. , we get our first desired.
\[
\Gamma(I) \subseteq Z(N)
\] (1)

For all \( x \in N \) and \( \alpha \in Γ \), then \( xγa \subseteq I \) and by Eq. 1, \( xγa \subseteq Z(N) \), also for all \( \alpha \in Γ \) and \( y \in N \)

Then
\[
0 = [y, xγa] = [y, x]_γa γa \quad \text{in general}
\] (2)

\[
[y, x]_γa = (0)
\]

Since I is left ideal and by Eq. 2 , then \( [y, x]_γa γI = (0) \), but N is prime Γ-ring and I non-zero ideal that means \( [y, x]_γa = 0 \), for all \( x, y \in N \) and \( \alpha \in Γ \).

**Corollary 2.7.** Suppose N be a prime Γ-ring of characteristic different from 2, with condition (*), I be an ideal of N and \( T : N \rightarrow N \) be a Jordan left centralizer on I, then N is commutative.

**Proof:** by Theorem 2.4. , then N contains a central ideal and by Lemma 2.6. then N is commutative Γ-ring.

**Corollary 2.8.** Suppose N be a prime Γ-ring of characteristic different from 2, with condition (*), if \( T : N \rightarrow N \) be a left centralizer on N, then T is centralizer on N.

**Proof:** for all \( a \in N \) and \( γ \in Γ \), \( T(ay) = T(a)γa \), by corollary 2.7. then N is commutative, for all \( a, b \in N \) and \( γ \in Γ \), \( T(ab) = T(bγa) = T(b)γa = aγT(b) \), that is T also right centralizer.

**References**